Fusion rules and boundary conditions in the $c = 0$ triplet model

2009 J. Phys. A: Math. Theor. 42 325403

(http://iopscience.iop.org/1751-8121/42/32/325403)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 325403 (43pp) doi:10.1088/1751-8113/42/32/325403

Fusion rules and boundary conditions in the $c = 0$ triplet model

Matthias R Gaberdiel[1], **Ingo Runkel**[2] **and Simon Wood**[1]

[1] Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland
[2] Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

E-mail: gaberdiel@itp.phys.ethz.ch, ingo.runkel@kcl.ac.uk and swood@itp.phys.ethz.ch

Received 14 May 2009
Published 27 July 2009
Online at stacks.iop.org/JPhysA/42/325403

Abstract

The logarithmic triplet model $\mathcal{W}_{2,3}$ at $c = 0$ is studied. In particular, we determine the fusion rules of the irreducible representations from first principles and show that there exists a finite set of representations, including all irreducible representations, that closes under fusion. With the help of these results, we then investigate the possible boundary conditions of the $\mathcal{W}_{2,3}$ theory. Unlike the familiar Cardy case where there is a consistent boundary condition for every representation of the chiral algebra, we find that for $\mathcal{W}_{2,3}$ only a subset of representations gives rise to consistent boundary conditions. These then have boundary spectra with non-degenerate two-point correlators.

PACS number: 11.25.Hf

Contents

1. Introduction and summary

Logarithmic conformal field theories appear in the description of critical points in many interesting physical systems. Some examples are polymers, spin chains, percolation and sand-pile models; see for example [1–9] for some recent papers. Logarithmic conformal field theories have also played an important role in recent attempts to understand chiral massive gravity [10, 11]. In some of these examples, in particular for critical systems with quenched disorder, for dilute self-avoiding polymers, and for percolation, as well as in the context of chiral gravity, the logarithmic conformal field theory has central charge $c = 0$; see e.g. [12] for a discussion of $c = 0$ theories. This has important consequences for the structure of the resulting theory. Indeed logarithmic conformal field theories at $c = 0$ behave rather differently from the examples that have been studied in detail so far, in particular from the $(1, p)$-series whose structure has now been largely understood [13–17].

In this paper we want to study one particular $c = 0$ logarithmic conformal field theory, namely the $\mathcal{W}_{2,3}$ triplet theory. This is the simplest example of a whole family of $\mathcal{W}_{p,q}$ triplet theories that can be naturally associated with the minimal models [18, 19]. One peculiar feature of the $\mathcal{W}_{2,3}$ model (and of all $\mathcal{W}_{p,q}$ theories with $p, q \geqslant 2$) is that the vacuum representation is not irreducible. This is a generic property of $c = 0$ logarithmic conformal field theories[3], and we believe that it is responsible for the complicated and rather unfamiliar behaviour we shall encounter.

Our first main result concerns the description of the $\mathcal{W}_{2,3}$ fusion rules of all indecomposable representations that appear as direct summands in successive fusions of the irreducibles. Our analysis starts from the corresponding Virasoro fusion rules which we re-examine following [20]. Using induced representations and associativity, we then determine the fusion rules of the irreducible $\mathcal{W}_{2,3}$-representations as well as those of the resulting indecomposable representations. Our results agree with [6, 21], but go beyond them in that we also determine the fusion rules of representations that are not accessible in their approach. Furthermore, we shall exhibit some of the unusual properties of these $\mathcal{W}$-representations and their fusion. For example, there is a subtle difference between 'conjugate' and 'dual' representations that we shall explain in some detail (see section 1.1.1), and the Grothendieck group (that appears naturally in the construction of the boundary theory) does not possess the standard ring structure; see section 1.1.3.

[3] For $c = 0$, the descendant of the vacuum, $L_{-2}\Omega$, is a Virasoro highest weight vector since $L_1 L_{-2}\Omega = L_2 L_{-2}\Omega = 0$. Unless the stress tensor of the conformal field theory vanishes, the vacuum representation is reducible as a representation of the Virasoro algebra. It may of course still be irreducible as a representation of a larger chiral algebra (for example, if one takes the product of two non-logarithmic theories with opposite central charge), but for $\mathcal{W}_{2,3}$, and in fact for all $\mathcal{W}_{p,q}$ with $p, q \geqslant 2$, the vacuum representation contains a non-trivial sub-representation of the entire $\mathcal{W}$-algebra.

The fusion rules are an important ingredient for the description of the possible boundary conditions. Boundary logarithmic conformal field theories have been investigated from several points of view, for example starting from an underlying lattice realization [1–3, 6, 9, 19, 22–27], from supergroup WZW models [28–33] or from logarithmic extensions of Virasoro minimal models [7, 16, 34–42]. The work of most direct relevance to our purposes is [6], where the fusion rules of the $\mathcal{W}_{2,3}$ model are analysed via the boundary theory (on a lattice) under the assumption that one can read off the fusion rules from the open string spectra as in Cardy's analysis [49]. Indeed, in the usual (non-logarithmic rational) case, there is a boundary condition for every representation of the chiral algebra, and the open string spectrum between two such boundary conditions agrees precisely with the fusion of the corresponding representations (or rather, the fusion where one of the two representation is replaced by its conjugate representation). For $\mathcal{W}_{2,3}$, on the other hand, not every representation corresponds to a consistent boundary condition.

More specifically, if we try to construct a boundary theory where all representations of $\mathcal{W}_{2,3}$ correspond to boundary conditions, it is possible to define an associative operator product expansion (OPE) of boundary fields, but the two-point correlator of boundary fields will in general be degenerate. The boundary conditions with a non-degenerate two-point correlator correspond essentially to representations whose conjugate representation agrees with the dual representation (for more details, see section 1.2.2). If $\mathcal{R}$ and $\mathcal{S}$ are two such representations, the open string spectrum between the corresponding boundary conditions is given by the fusion of $\mathcal{R}$ with the conjugate representation of $\mathcal{S}$, just as in Cardy's analysis of the non-logarithmic case. This is the second main result of our paper, and it reproduces precisely the lattice results of [6] from an analysis intrinsic to conformal field theory.

In non-logarithmic rational conformal field theories one can uniquely reconstruct the bulk theory from a consistent boundary theory [43–46], and every possible bulk theory (with the appropriate symmetry algebra) can be obtained in this way [47]. Furthermore, two boundary theories give rise to isomorphic bulk theories if and only if the boundary theories are equivalent in the sense described in [48]. The boundary theory is typically simpler than the bulk theory, and it is therefore often useful to start with the boundary theory in order to construct the bulk theory. This is most pronounced in the charge-conjugation Cardy case [49], where there is a boundary condition whose open string spectrum consists just of the vacuum representation of the chiral algebra.

One may hope that the general idea—to start from a boundary theory in order to construct the bulk theory that fits to it—also remains valid in the logarithmic case, even if the detailed construction will start to deviate. For $\mathcal{W}_{1,p}$ models this approach was used in [16] to obtain a modular invariant bulk partition function, which for $p = 2$ reproduced the known local theory from [50]. This analysis was performed for the analogue of the Cardy case, i.e. by starting with a boundary condition whose open string spectrum consists just of the vacuum representation. However, for the $\mathcal{W}_{2,3}$ theory, such a boundary condition does not exist since the corresponding boundary two-point correlators are degenerate. This suggests that the analogue of the charge-conjugation modular invariant for the $\mathcal{W}_{2,3}$ model will be more involved than for the $\mathcal{W}_{1,p}$ series [16]. Nonetheless, because a lattice realization of the $\mathcal{W}_{2,3}$ theory is known [6], it seems plausible that a consistent bulk theory does in fact exist. Furthermore, there is a fairly natural guess for how the construction of the bulk theory could roughly work; this is briefly indicated in section 4.

In the remainder of this introduction, we give a detailed (but non-technical) overview of the results of the paper. Section 2 contains the detailed discussion of the $\mathcal{W}$-representations and the computation of their fusion products. In section 3 we construct the boundary theory based on the abstract theory of internal Homs and dual objects in tensor categories, and section 4 contains our conclusions. In appendix A, we list the characters of the $\mathcal{W}_{2,3}$-representations, their embedding diagrams and their fusion rules. We also spell out a dictionary of our notation and that of [6, 21]; see appendix A.2. Finally, appendix B contains some technicalities needed in section 3.

1.1. $\mathcal{W}$-representations and fusion rules

Let us begin by reviewing the structure of the underlying Virasoro theory. Recall that the Virasoro minimal models have central charge

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \tag{1.1}$$

where p and q are a pair of positive coprime integers. The vacuum representation is the irreducible representation based on the highest weight state Ω with $h = 0$. The corresponding Verma module has two independent null vectors: the null vector $\mathcal{N}_1 = L_{-1}\Omega$ of conformal dimension $h = 1$ and a null vector $\mathcal{N}_2$ of conformal dimension $h = (p-1)\cdot(q-1)$. Setting $\mathcal{N}_1$ and $\mathcal{N}_2$ to zero, we obtain the irreducible vacuum representation based on Ω. The highest weight representations of the corresponding vertex operator algebra are the representations of the Virasoro algebra for which the modes $V_n(\mathcal{N}_1)$ and $V_n(\mathcal{N}_2)$ act trivially. They have conformal weights

$$h_{r,s} = \frac{(ps-qr)^2-(p-q)^2}{4pq}, \tag{1.2}$$

where $1 \leqslant r \leqslant p-1$, $1 \leqslant s \leqslant q-1$ and we have the identification

$$h_{r,s} = h_{p-r,q-s}. \tag{1.3}$$

We shall be mainly interested in the case $(p, q) = (2, 3)$ for which $c_{2,3} = 0$. In this case, the null vector $\mathcal{N}_2$ of the vacuum representation is just the vector $\mathcal{N}_2 = L_{-2}\Omega$, and thus the irreducible vacuum representation $\mathcal{V}(0)$ only consists of the vacuum state Ω itself. Furthermore, there is only one representation in (1.2), namely the vacuum representation $\mathcal{V}(0)$ itself. This is clearly a very trivial and boring theory.

The logarithmic theory we are interested in is obtained in a slightly different fashion. Instead of taking the vertex operator algebra to be $\mathcal{V}(0)$, we consider the vertex operator algebra $\mathcal{V}$ that is obtained from the Verma module based on Ω by dividing out $\mathcal{N}_1 = L_{-1}\Omega$, but not $\mathcal{N}_2 \equiv T = L_{-2}\Omega$. This leads to a logarithmic conformal field theory, but not to one that is rational. In order to make the theory rational, we then enlarge the chiral algebra by three fields of conformal dimension 15. The resulting vertex operator algebra will be denoted by $\mathcal{W}_{2,3}$ or just $\mathcal{W}$, and it defines the so-called $\mathcal{W}_{2,3}$ model [18]. Its irreducible representations are described by the finite Kac table:

$$\begin{array}{c|ccc} & s=1 & s=2 & s=3 \\ \hline r=1 & 0,2,7 & 0,1,5 & \frac{1}{3},\frac{10}{3} \\ r=2 & \frac{5}{8},\frac{33}{8} & \frac{1}{8},\frac{21}{8} & -\frac{1}{24},\frac{35}{24} \end{array} \tag{1.4}$$

Here each entry h is the conformal dimension of the highest weight states of an irreducible representation, which we shall denote by $\mathcal{W}(h)$. There is only one representation corresponding to $h = 0$, namely the one-dimensional vacuum representation $\mathcal{W}(0)$, spanned by the vacuum vector Ω.

The representations $\mathcal{W}(h)$ for which the value h is coloured grey in (1.4) will not correspond to consistent boundary conditions; see section 1.2.2.

As is familiar from other logarithmic theories, the 13 irreducible representations in (1.4) do not close among themselves under fusion. However, one can show that the fusion rules close on some larger set, involving in addition 22 indecomposable representations. These will be described in more detail in section 2, and their characters will be given in appendix A.1; the relation to the notation in [6, 21] is explained in appendix A.2.

$$\begin{aligned}
&\mathcal{W},\ \mathcal{W}^*,\ \mathcal{Q},\ \mathcal{Q}^*,\ \mathcal{R}^{(2)}(0,2)_7, \mathcal{R}^{(2)}(2,7), \mathcal{R}^{(2)}(0,1)_5, \mathcal{R}^{(2)}(1,5),\\
&\mathcal{R}^{(2)}(0,2)_5, \mathcal{R}^{(2)}(2,5), \mathcal{R}^{(2)}(0,1)_7, \mathcal{R}^{(2)}(1,7), \mathcal{R}^{(2)}\left(\tfrac{1}{3},\tfrac{1}{3}\right),\\
&\mathcal{R}^{(2)}\left(\tfrac{1}{3},\tfrac{10}{3}\right), \mathcal{R}^{(2)}\left(\tfrac{5}{8},\tfrac{5}{8}\right), \mathcal{R}^{(2)}\left(\tfrac{5}{8},\tfrac{21}{8}\right), \mathcal{R}^{(2)}\left(\tfrac{1}{8},\tfrac{1}{8}\right), \mathcal{R}^{(2)}\left(\tfrac{1}{8},\tfrac{33}{8}\right),\\
&\mathcal{R}^{(3)}(0,0,1,1), \mathcal{R}^{(3)}(0,0,2,2), \mathcal{R}^{(3)}(0,1,2,5), \mathcal{R}^{(3)}(0,1,2,7).
\end{aligned} \tag{1.5}$$

Again, the representations whose names are coloured grey will not correspond to consistent boundary conditions. We do not claim that (1.4) and (1.5) are *all* indecomposable representations of $\mathcal{W}_{2,3}$. Indeed it is clear that they are not, as they are not closed under taking quotients and sub-representations. However, the representations (1.4) and (1.5) form the minimal set of representations, containing the irreducible representations in (1.4), that closes under fusion and taking conjugates.

Since the vertex operator algebra $\mathcal{W}$ contains generating fields at the rather high conformal weight $h = 15$, it is difficult to determine the commutation relations of this $\mathcal{W}$-algebra explicitly, and thus we do not know how to determine the fusion rules directly[4]. However, we can infer the $\mathcal{W}$ fusion rules from the calculation of the fusion rules of the Virasoro vertex operator algebra $\mathcal{V}$, using induced representations. The $\mathcal{V}$ fusion rules, on the other hand, can be determined explicitly, using the techniques of [52, 53] (see section 2). In fact, this analysis has already been done some time ago by Eberle and Flohr [20], but it contained a small mistake which we have corrected here. The resulting $\mathcal{V}$ fusion rules are associative and commutative, and the same then also holds for the induced $\mathcal{W}$ fusion rules. We list all fusion products for the representations in (1.4) and (1.5) in appendix A.4

The resulting fusion rules are much more complicated than for example those of the well-understood logarithmic $(1, p)$ models. The source of this and many other difficulties is probably the fact that the vertex operator algebra $\mathcal{W}$ is not irreducible. In fact, $\mathcal{W}$ does not agree with the irreducible representation $\mathcal{W}(0)$ based on Ω, since in $\mathcal{W}$ the state $T = L_{-2}\Omega$ does not vanish, but generates the proper sub-representation $\mathcal{W}(2) \subset \mathcal{W}$. The structure of $\mathcal{W}$ is thus described by the embedding diagram

$$\begin{tikzcd}
\Omega \arrow[r, dotted] \arrow[rr, bend left] & \times & T \arrow[r, "\mathcal{W}(2)"] & {} \\
h=0 & h=1 & h=2 &
\end{tikzcd} \tag{1.6}$$

where the arrows describe the action of the $\mathcal{W}$-modes and '$\times$' refers to the null vector $\mathcal{N}_1$ which has been divided out. Alternatively, we can characterize $\mathcal{W}$ by the exact sequence

$$0 \longrightarrow \mathcal{W}(2) \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}(0) \longrightarrow 0. \tag{1.7}$$

In the following, we shall summarize some of the rather peculiar features of the resulting theory and its fusion rules.

1.1.1. Conjugate and dual representations. The conjugate representation $\mathcal{R}^*$ of a representation $\mathcal{R}$ is characterized by the property that the two-point conformal blocks involving one state from $\mathcal{R}$ and one state from $\mathcal{R}^*$ define a non-degenerate bilinear form on $\mathcal{R} \times \mathcal{R}^*$. Usually, the vertex operator algebra itself is self-conjugate since the vacuum state Ω is a self-conjugate state, and the vertex operator algebra is irreducible. However, in the present

[4] One may hope that the description given in [51] may allow one to overcome this limitation.

case, the latter property does not hold, and as a consequence $\mathcal{W}^*$ is not isomorphic to $\mathcal{W}$. In fact, $\mathcal{W}^*$ is characterized by the exact sequence

$$0 \longrightarrow \mathcal{W}(0) \longrightarrow \mathcal{W}^* \longrightarrow \mathcal{W}(2) \longrightarrow 0, \tag{1.8}$$

and is therefore different from $\mathcal{W}$. It is generated from a state t at conformal weight $h = 2$:

$$\begin{tikzcd} \omega \arrow[r, dotted] & \times & t \arrow[ll, bend right] \arrow[r, "\mathcal{W}(2)"] & {} \\ h=0 & h=1 & h=2 & \end{tikzcd} \tag{1.9}$$

but t is not a highest weight state since $L_2 t = \omega$. On the other hand, ω is annihilated by all $\mathcal{W}$ modes. The fact that $\mathcal{W}$ is not self-conjugate means that, amongst other things, $\mathcal{W}$ cannot appear by itself as the open string spectrum of a boundary condition—this will be explained in more detail below.

In non-logarithmic rational conformal field theories, the fusion $\mathcal{R} \otimes \mathcal{R}^*$ always contains the vertex operator algebra $\mathcal{W}$ itself. Thus, it makes sense to call the conjugate representation $\mathcal{R}^*$ also the 'dual representation'. Furthermore, it is then obvious that the fusion of $\mathcal{R}$ with $\mathcal{R}\otimes\mathcal{R}^*$ contains $\mathcal{R}$. These properties motivate an abstract categorical definition of duals which we review in section 3.4. Two necessary conditions for the existence of a dual representation $\mathcal{R}^\vee$ are that there exist non-zero intertwiners

$$b_{\mathcal{R}} : \mathcal{W} \to \mathcal{R} \otimes \mathcal{R}^\vee \qquad \text{and} \qquad d_{\mathcal{R}} : \mathcal{R}^\vee \otimes \mathcal{R} \to \mathcal{W} \tag{1.10}$$

and the image of $b_{\mathcal{R}}$ in $\mathcal{R} \otimes \mathcal{R}^\vee$ should not give zero when fused with either $\mathcal{R}$ or $\mathcal{R}^\vee$. In the general logarithmic case, the conjugate representation $\mathcal{R}^*$ does not automatically satisfy these properties and thus the dual representation $\mathcal{R}^\vee$ may not agree with the conjugate representation $\mathcal{R}^*$ (or may not even exist at all). For example, each of the irreducible representations $\mathcal{W}(h)$ is self-conjugate, $\mathcal{W}(h)^* = \mathcal{W}(h)$, but for $\mathcal{W}(0)$ and $\mathcal{W}(2)$ we have the fusions

$$\mathcal{W}(0) \otimes \mathcal{W}(0) = \mathcal{W}(0), \qquad \mathcal{W}(2) \otimes \mathcal{W}(2) = \mathcal{W}^*. \tag{1.11}$$

Thus, $\mathcal{W}(0)$ is not self-dual since $d_{\mathcal{W}(0)}$ is zero—there is simply no non-zero intertwiner from $\mathcal{W}(0)$ to $\mathcal{W}$. Furthermore, because any intertwiner from $\mathcal{W}$ to $\mathcal{W}^*$ has to factor through $\mathcal{W}(0)$, the image of $b_{\mathcal{W}(2)}$ is contained in $\mathcal{W}(0) \subset \mathcal{W}(2) \otimes \mathcal{W}(2)$. But $\mathcal{W}(0) \otimes \mathcal{W}(2) = 0$ and so $\mathcal{W}(2)$ is not self-dual either. In fact, neither $\mathcal{W}(0)$ nor $\mathcal{W}(2)$ have a dual representation at all. The same also holds for $\mathcal{W}(1)$, $\mathcal{W}(5)$ and $\mathcal{W}(7)$.

We believe that the indecomposable representations listed in (1.4) and (1.5) which are not in grey boxes are all self-dual and self-conjugate; see appendix A.3. In particular, for these representations the conjugates agree with the duals.

1.1.2. Exactness of the fusion product. Another strange feature of the $\mathcal{W}_{2,3}$ theory is that the fusion product is not exact, i.e. fusion does not in general respect exact sequences. Indeed, if we consider the fusion of each entry of (1.8) with $\mathcal{W}(0)$, using the fusion rules (1.11) as well as

$$\mathcal{W}(0) \otimes \mathcal{W}^* = 0 \qquad \text{and} \qquad \mathcal{W}(0) \otimes \mathcal{W}(h) = 0 \quad \text{for} \quad h \neq 0, \tag{1.12}$$

we get the sequence $0 \longrightarrow \mathcal{W}(0) \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ which is clearly not exact. However, the fusion rules we have determined appear to be right-exact, i.e. the last three entries of an exact sequence are mapped to an exact sequence under fusion.

1.1.3. The Grothendieck group. The Grothendieck group $\mathrm{K}_0 \equiv \mathrm{K}_0(\mathrm{Rep}(\mathcal{W}))$ of the category of representations of $\mathcal{W}$ is, roughly speaking, the quotient set obtained by identifying two representations if they have the same character[5]. Let us denote the equivalence class of a representation $\mathcal{R}$ by $[\mathcal{R}]$. The group operation is addition, defined via the direct sum of representations,

$$[\mathcal{R}] + [\mathcal{R}'] = [\mathcal{R} \oplus \mathcal{R}']. \tag{1.13}$$

For example, given the exact sequence (1.8), we have

$$[\mathcal{W}^*] = [\mathcal{W}(0) \oplus \mathcal{W}(2)] = [\mathcal{W}(0)] + [\mathcal{W}(2)] \tag{1.14}$$

since (1.8) implies that the characters obey $\chi_{\mathcal{W}^*} = \chi_{\mathcal{W}(0)} + \chi_{\mathcal{W}(2)}$. If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are the irreducible representations, one can convince oneself that the Grothendieck group is the free Abelian group generated by $[\mathcal{R}_1], \ldots, [\mathcal{R}_n]$, i.e. $\mathrm{K}_0 = \mathbb{Z}[\mathcal{R}_1] \oplus \cdots \oplus \mathbb{Z}[\mathcal{R}_n]$. In other words, the elements of K_0 are all linear combinations of $[\mathcal{W}(h)]$ with integer coefficients, where h takes one of the 13 values from the Kac table in (1.4).

For non-logarithmic rational conformal field theories, the Grothendieck group also has a product structure which is defined by

$$[\mathcal{R}] \cdot [\mathcal{S}] = [\mathcal{R} \otimes \mathcal{S}]. \tag{1.15}$$

The physical significance of this product is that the character associated with $[\mathcal{R}] \cdot [\mathcal{S}^*]$ is precisely the character of the open string spectrum between the Cardy boundary conditions $\mathcal{R}$ and $\mathcal{S}$ [49]. Furthermore, the structure constants of the multiplication (1.15) are determined by the Verlinde formula. A similar structure also appears for the $\mathcal{W}_{1,p}$ models [14, 16, 54].

As we have mentioned before, the construction of the boundary conditions is more subtle in the $\mathcal{W}_{2,3}$ model. This is reflected by the fact that the product (1.15) is actually not well defined on K_0. To see this, we observe that we can compute $[\mathcal{W}(0)] \cdot [\mathcal{W}^*]$ in two ways:

$$\begin{aligned}
&[\mathcal{W}(0)] \cdot [\mathcal{W}^*] = [\mathcal{W}(0) \otimes \mathcal{W}^*] = 0 \text{ versus} \\
&[\mathcal{W}(0)] \cdot [\mathcal{W}^*] = [\mathcal{W}(0)] \cdot ([\mathcal{W}(0)] + [\mathcal{W}(2)]) \\
&\qquad = [\mathcal{W}(0) \otimes \mathcal{W}(0)] + [\mathcal{W}(0) \otimes \mathcal{W}(2)] = [\mathcal{W}(0)].
\end{aligned} \tag{1.16}$$

Thus for $\mathcal{W}_{2,3}$, the fusion of representations does not induce a product on the Grothendieck group. However, we can restrict ourselves to the subset of those representations that correspond to consistent boundary conditions, and on this subset it is in fact possible to define product (1.15)—this will be explained in more detail in section 1.2.3.

For completeness, we also mention that one can define the multiplication (1.15) on the quotient

$$\widetilde{\mathrm{K}}_0 = \mathrm{K}_0/(\mathbb{Z}[\mathcal{W}(0)]). \tag{1.17}$$

This amounts to setting $[\mathcal{W}(0)]$ to zero. The resulting ring structure on the quotient $\widetilde{\mathrm{K}}_0$ coincides with the one described in [18, section 6.3] using quantum groups (we explain the relation to the notation of [18] in appendix A.2).

1.1.4. Properties of the category $\mathrm{Rep}(\mathcal{W})$. It is instructive to summarize the properties of the representation category of the $\mathcal{W}_{2,3}$ vertex operator algebra and compare them to those of the usual non-logarithmic Virasoro minimal models $\mathcal{V}_{p,q}$ and the logarithmic $\mathcal{W}_{1,p}$ models. In the following table, $\mathcal{V}$ is the vertex operator algebra, the ticks '✓' are results which have been

[5] This is true for $\mathcal{W}_{2,3}$ and whenever the characters of all irreducible representations are linearly independent. We recall the general definition in section 3.5.

proved, the ticks in brackets '(✓)' are supported by evidence but not proved and the negative results '–' are proved by counter-example.

		$\mathcal{V} = \mathcal{V}_{p,q}$	$\mathcal{V} = \mathcal{W}_{1,p}$	$\mathcal{V} = \mathcal{W}_{2,3}$
(1)	L_0 diagonalizable on $\mathcal{V}$	✓	✓	✓
(2)	$\mathrm{End}(\mathcal{V}) = \mathbb{C}\mathrm{id}_{\mathcal{V}}$	✓	✓	✓
(3)	$\mathcal{V}$ irreducible	✓	✓	–
(4)	$\mathrm{Rep}(\mathcal{V})$ is a braided tensor category	✓	✓	(✓)
(5)	$\mathrm{Rep}(\mathcal{V})$ has duals	✓	(✓)	–
(6)	Tensor product is right-exact	✓	(✓)	(✓)
(7)	Tensor product is exact	✓	(✓)	–
(8)	Tensor product induces a product on K_0	✓	(✓)	–
(9)	$\mathcal{V}$ has a finite number of irreducibles	✓	✓	(✓)
(10)	$\mathrm{Rep}(\mathcal{V})$ is semi-simple	✓	–	–

A few comments are maybe in order: $\mathrm{End}(\mathcal{V})$ in (2) describes the space of $\mathcal{V}$-intertwiners from $\mathcal{V}$ to itself, and in all three cases this just consists of multiplication by complex numbers. This is automatic if $\mathcal{V}$ is irreducible, but it is also true for $\mathcal{W}_{2,3}$ since the intertwiner is uniquely determined by its action on the cyclic vector Ω, and because the L_0-eigenspace of eigenvalue 0 is one dimensional, it can only map Ω to a multiple of itself. In category-speak, this means that $\mathcal{W}_{2,3}$ is absolutely simple but not simple.

Let us give some references to the literature where the results in the above table can be found.

$\mathcal{V}_{p,q}$: (1)–(3) hold by construction, and for (9) and (10) see [55, definition 2.3 and theorem 4.2]. The existence of a braiding[6] and tensor product follows from [58, theorem 3.10], and the duality morphisms are constructed in [59, theorem 3.8], establishing (4) and (5). The existence of duals implies that the tensor product is exact (see [57, proposition 2.1.8]), which in turn guarantees that the product on K_0 is well defined, so that (6)–(8) hold as well.

$\mathcal{W}_{1,p}$: (1)–(3) again hold by construction; see [60] and the free field approach in [14]. A tensor product theory for vertex operator algebras with logarithmic intertwiners has been developed in [61]. By [62, proposition 4.1], the theory can be applied for vertex operator algebras which are C_2-cofinite and of positive energy. By [15, 17] the $\mathcal{W}_{1,p}$-vertex operator algebras are of this type, and as a consequence also satisfy (9). It follows that $\mathrm{Rep}(\mathcal{W}_{1,p})$, defined as in [62, proposition 4.3], is a braided tensor category [62, theorem 4.11]. This establishes point (4). That (10) does not hold can be seen for example from [14, section 2.4] or [17, section 4]. Finally, (5) (and consequently (6)–(8)) would follow from [63, conjecture 4.2].

$\mathcal{W}_{2,3}$: The explicit free field construction of [18, definition 4.1 and theorem 4.2] establishes (1) and (2). It also shows that (3) and (10) do not hold. Counter-examples to (5), (7) and (8) were provided in sections 1.1.1–1.1.3, respectively. As far as we know, it has not been proven that $\mathcal{W}_{2,3}$ is C_2-cofinite and that the tensor product theory of [61] can be applied to this model. However, the results of [6, 21] and the fusion rule computations of section 2 strongly support (4). The tests we have for (6) are far less stringent, but it

[6] In all three cases of the above table, the map $\mathrm{e}^{-2\pi \mathrm{i} L_0}$ should endow the representation category with a twist in the sense of [56, definition 6.1]. Alternatively, the twist can be introduced as a morphism derived from a functorial isomorphism from a representation to its double-dual as in [57, section 2.2]. The latter formulation requires the existence of duals. As far as we can tell, in the vertex operator algebra literature the question of the existence of a twist and of duals has not been addressed separately, and so we have omitted the twist as a separate property from the table.

is certainly a natural property to expect. Finally, in favour of (9) we observe that there are only a finite number of irreducible representations which can be obtained via the free field construction of [18, section 4.3] and that no additional irreducible representations appear in our computation of fusion products.

1.2. Boundary conditions and open string spectra

With this detailed understanding of the fusion rules, we can now turn to describing the possible boundary conditions, their open string spectra and the OPEs of the corresponding boundary fields. We shall only consider boundary conditions that preserve the $\mathcal{W}$ symmetry. Let us first explain more precisely what we mean by this—since we want to construct a consistent boundary theory without having to specify the bulk theory first, this is somewhat subtle.

1.2.1. $\mathcal{W}$-symmetric boundary conditions. In order to speak of a $\mathcal{W}$-symmetric boundary condition, we implicitly assume that the corresponding bulk theory has the symmetry $\mathcal{W}_{\text{left}} \otimes_{\mathbb{C}} \mathcal{W}_{\text{right}}$, with $\mathcal{W}_{\text{left}} = \mathcal{W}_{\text{right}} = \mathcal{W}$. In other words, there is an inclusion of $\mathcal{W}_{\text{left}} \otimes_{\mathbb{C}} \mathcal{W}_{\text{right}}$ into the space of states of the bulk theory $\mathcal{H}_{\text{bulk}}$ which respects operator products. This then turns $\mathcal{H}_{\text{bulk}}$ into a representation of $\mathcal{W}_{\text{left}} \otimes_{\mathbb{C}} \mathcal{W}_{\text{right}}$. For logarithmic conformal field theories, the inclusion of $\mathcal{W}_{\text{left}} \otimes_{\mathbb{C}} \mathcal{W}_{\text{right}}$ need not be a direct summand of $\mathcal{H}_{\text{bulk}}$; the $\mathcal{W}_{1,p}$ models provide an example of this [16, 50].

Suppose we consider such a conformal field theory on the upper half-plane with a boundary condition on the real line labelled by A. For A to be a $\mathcal{W}$-symmetric boundary condition, we demand that on the real line the fields of the left- and right-moving copies of $\mathcal{W}$ are related by $W_{\text{left}}(x) = W_{\text{right}}(x)$, where $W \in \mathcal{W}$ and $W_{\text{left}} = W \otimes_{\mathbb{C}} \Omega$, $W_{\text{right}} = \Omega \otimes_{\mathbb{C}} W$ [64]. This implies, in particular, that there is a map $\eta_A : \mathcal{W} \to \mathcal{H}_{A\to A}$ and that the boundary fields $\mathcal{H}_{A\to A}$ on A form a representation of $\mathcal{W}$. Similarly, the spaces $\mathcal{H}_{A\to B}$ of boundary changing fields between two $\mathcal{W}$-symmetric boundary conditions A and B are $\mathcal{W}$-representations.

Given two (not necessarily different) boundary conditions A and B, we require that the two point correlators of boundary (changing) fields are non-degenerate. Otherwise, if, say, a field $\psi \in \mathcal{H}_{A\to B}$ had a zero two-point correlator with all fields in $\mathcal{H}_{B\to A}$, then ψ would vanish in all correlation functions and we should replace $\mathcal{H}_{A\to B}$ with its quotient by the kernel of the two-point correlator. We assume that this has been done, and so all two-point correlators are non-degenerate. The two-point correlators themselves are determined by the OPE of boundary fields and their one-point correlators. We describe the one-point correlator on a boundary with label A by a $\mathcal{W}$-intertwiner $\varepsilon_A : \mathcal{H}_{A\to A} \to \mathcal{W}^*$. The reason to take the image of ε_A to be $\mathcal{W}^*$ rather than $\mathbb{C}$ is that this can be more directly translated into a condition defined in the category $\text{Rep}(\mathcal{W})$. The interpretation is that each boundary field gives rise to a linear functional on $\mathcal{W}$ by placing the boundary field at 0 and a field in $\mathcal{W}$ at ∞, using the embedding η_A. The one-point correlator itself is obtained by placing the vacuum Ω at ∞.

We shall also demand that $\eta_A : \mathcal{W} \to \mathcal{H}_{A\to A}$ is *injective*. For suppose $\mathcal{N} \subset \mathcal{W}$ is annihilated by η_A. The sewing constraint arising from the two-point correlator on the upper half-plane [65, 66] shows that a correlator on the upper half-plane which involves at least one field from $\mathcal{N}$ has to vanish. It follows that $\mathcal{N}$ is an ideal in $\mathcal{W}$ and that $\mathcal{N}$-descendents in $\mathcal{H}_{\text{bulk}}$ act as zero. We therefore no longer describe a conformal field theory with $\mathcal{W}$-symmetry, but a conformal field theory with $\mathcal{W}/\mathcal{N}$-symmetry in the presence of a $\mathcal{W}/\mathcal{N}$-symmetric boundary condition. In particular, the bulk theory reconstructed with the procedure of [16] may then depend on the boundary condition we start from.

Summarizing the above discussion, a consistent $\mathcal{W}$-symmetric boundary theory thus consists of the following data:

– a collection $\mathcal{B} = \{A, B, \ldots\}$ of labels for boundary conditions,
– for each pair of labels A, B an open string spectrum $\mathcal{H}_{A\to B}$ which is a $\mathcal{W}$-representation,
– a boundary OPE $\mathcal{H}_{B\to C} \times \mathcal{H}_{A\to B} \to \mathcal{H}_{A\to C}$ compatible with the $\mathcal{W}$-symmetry,
– $\mathcal{W}$-intertwiners $\eta_A : \mathcal{W} \to \mathcal{H}_{A\to A}$,
– one-point correlators $\varepsilon_A : \mathcal{H}_{A\to A} \to \mathcal{W}^*$ which are $\mathcal{W}$-intertwiners.

These data should satisfy the following sewing constraints and non-degeneracy conditions:

(B1) The boundary OPE is associative.
(B2) The two-point correlator obtained by taking the OPE of two boundary fields and evaluating with ε_A is non-degenerate.
(B3) η_A is injective and $\eta_A(\Omega)$ is the identity field on the A-boundary.

Conditions B1 and B2 are certainly necessary if we want a consistent theory whose states are distinguishable in correlators. Condition B3 has a different status, because dropping it does not lead to inconsistencies of the boundary theory. We impose it in our analysis for the reason outlined above.

For a non-logarithmic rational vertex operator algebra $\mathcal{V}$, there is a canonical boundary theory [43, 44, 49, 67]: $\mathcal{B}$ consists of all $\mathcal{V}$-representations, $\mathcal{H}_{A\to B} = B \otimes A^*$ and the OPE can be defined using the duality intertwiners d_A (cf section 3.4). The same is true for the logarithmic rational $\mathcal{W}_{1,p}$ models [16, 40]. For the $\mathcal{W}_{2,3}$ model this ansatz turns out to work as well, but with one crucial difference: we can no longer assign consistent boundary conditions to *all* $\mathcal{W}$-representations, but only to a subset, as we will illustrate now.

1.2.2. A boundary theory for the $\mathcal{W}_{2,3}$ model. Given two representations A and B, there is a general categorical construction, called the 'internal Hom' $[A, B]$ (see section 3.3), which is the natural candidate for the open string spectrum, $\mathcal{H}_{A\to B} = [A, B]$. The reason for this proposal is that the internal Hom construction provides us with an associative boundary OPE, i.e. B1 is automatically satisfied.

Actually, as also explained in section 3.3, we can always express the internal Hom as

$$[A, B] = (A \otimes B^*)^*, \tag{1.18}$$

where A^* is the conjugate representation to A. In order to see that this proposal for the boundary spectrum is not so unnatural, consider the case $A = B = \mathcal{W}(2)$. Recall that $\mathcal{W}(2)^* \cong \mathcal{W}(2)$ and $\mathcal{W}(2) \otimes \mathcal{W}(2) = \mathcal{W}^*$ so that $\mathcal{H}_{\mathcal{W}(2)\to\mathcal{W}(2)} = [\mathcal{W}(2), \mathcal{W}(2)] = \mathcal{W}$. Had we taken $\mathcal{H}_{A\to B} = B \otimes A^*$ as in the non-logarithmic case, the result for $\mathcal{H}_{\mathcal{W}(2)\to\mathcal{W}(2)}$ would have been $\mathcal{W}^*$ which is different from $\mathcal{W}$ and does not allow for a unit $\eta : \mathcal{W} \to \mathcal{W}^*$ (because this would factor through $\mathcal{W}(0)$ and $\mathcal{W}(0) \otimes \mathcal{W}^* = 0$). To summarize,

$$\mathcal{B}_{\text{first try}} = \{\text{all } \mathcal{W}\text{-representations}\} \text{ satisfies B1}. \tag{1.19}$$

However, it turns out that this attempt violates B2 and B3. To see that B2 fails consider $A = B = \mathcal{W}(2)$, for which we have just seen that $\mathcal{H}_{\mathcal{W}(2)\to\mathcal{W}(2)} = \mathcal{W}$. Recall from section 1.1.1 that a boundary condition with self-spectrum $\mathcal{W}$ does not allow for a non-degenerate two-point correlator, irrespective of what we choose for ε, simply because $\mathcal{W} \ncong \mathcal{W}^*$. B3 fails for $A = B = \mathcal{W}(0)$ because in this case $\mathcal{H}_{\mathcal{W}(0)\to\mathcal{W}(0)} = \mathcal{W}(0)$, and while there is an intertwiner $\eta_{\mathcal{W}(0)} : \mathcal{W} \to \mathcal{W}(0)$, it is not injective.

The obvious method to circumvent these problems is to remove all boundary labels from (1.19) for which B2 and B3 fail. A necessary condition for B3 to hold is that there exists an injective intertwiner $\mathcal{W} \to \mathcal{H}_{A\to A}$. We have just seen that this eliminates $\mathcal{W}(0)$, and in fact this is the only indecomposable representation ruled out by this criterion. A necessary condition for B2 to hold is that for any boundary label A, we have $(A \otimes A^*)^* \cong A \otimes A^*$.

This eliminates the irreducible representations $\mathcal{W}(1), \mathcal{W}(2), \mathcal{W}(5), \mathcal{W}(7)$, as well as the indecomposable representations $\mathcal{W}, \mathcal{W}^*, \mathcal{Q}, \mathcal{Q}^*$. Coming from the opposite direction, we will prove in section 3.4 that the following holds:

$$\mathcal{B} = \left\{ \begin{array}{l} \text{all } \mathcal{W}\text{-representations } A \text{ for which} \\ A^* \text{ is a dual for } A \text{ such that } b_A \text{ is injective} \end{array} \right\} \quad \text{satisfies B1–B3.} \tag{1.20}$$

Here b_A is the duality morphism mentioned in (1.10); it will serve to construct the unit η_A. We believe (but we have no proof) that of the 35 indecomposable $\mathcal{W}_{2,3}$-representations we consider in this paper, namely those listed in (1.4) and (1.5), only the 26 representations that are not written in a grey box are in $\mathcal{B}$.

These 26 indecomposable representations agree precisely with the boundary conditions considered in [6]. There, the boundary conditions were found by analysing a lattice model on a strip, while we obtain the list by representation theoretic arguments intrinsic to conformal field theory.

We will prove in section 3.4 that if $A, B \in \mathcal{B}$, then $(A \otimes B^*)^* \cong B \otimes A^*$. The open string spectra thus take the same form as in the non-logarithmic case,

$$\mathcal{H}_{A\to B} = B \otimes A^* \qquad \text{for} \quad A, B \in \mathcal{B}. \tag{1.21}$$

The construction of the boundary theory is then completely analogous to non-logarithmic rational conformal field theories and the $\mathcal{W}_{1,p}$ models; we provide the details in section 3.

We also prove (see theorem 3.10) that the space $\mathcal{H}_{A\to B}$ is always non-zero, i.e. there is a non-trivial spectrum of open strings between any two boundary conditions in $\mathcal{B}$. This is not true if for example the representation $\mathcal{W}(0)$ would be an allowed boundary condition. $\mathcal{W}(0)$ still satisfies B1 and B2 but not B3. Indeed, the spectrum of open strings between $\mathcal{W}(0)$ and $\mathcal{W}(2)$ would be $\mathcal{H}_{\mathcal{W}(0)\to\mathcal{W}(2)} = (\mathcal{W}(0) \otimes \mathcal{W}(2)^*)^* = 0$ because $\mathcal{W}(0) \otimes \mathcal{W}(2) = 0$.

1.2.3. Cylinder partition functions. As was already alluded to in section 1.1.3, the product structure of the Grothendieck group of a rational non-logarithmic theory is closely related to the cylinder diagram between two Cardy boundary conditions A and B:

$$Z(q)_{A\to B} = \mathrm{tr}_{\mathcal{H}_{A\to B}}(q^{L_0 - c/24}), \tag{1.22}$$

where $q = \exp(2\pi i\tau)$. Indeed, for Cardy boundary conditions, the open string spectrum is described by $B \otimes A^*$, and the character of $B \otimes A^*$ only depends on the class $[B \otimes A^*]$ in the Grothendieck group. As we have seen above (1.21), for boundary conditions labelled by $A, B \in \mathcal{B}$, the open string spectrum is still given by $B \otimes A^*$, and the character again only depends on $[B \otimes A^*]$. One may therefore expect that there should be a consistent product on the subgroup of the Grothendieck group that comes from the consistent boundary conditions. With this in mind, we introduce the subgroup of K_0 defined by

$$\mathrm{K}_0^b = (\text{subgroup generated by } [\mathcal{R}] \text{ for all } \mathcal{R} \in \mathcal{B} \text{ as defined in (1.20)}). \tag{1.23}$$

We shall give an explicit description of K_0^b in section 2.4, and we shall show in section 3.5 that on K_0^b the product $[\mathcal{R}] \cdot [\mathcal{R}'] := [\mathcal{R} \otimes \mathcal{R}']$ is indeed well defined and associative.

The fact that the product is well defined now implies that two boundary conditions $A, A' \in \mathcal{B}$ for which $[A] = [A']$ cannot be distinguished in any cylinder partition function,

$$A, A' \in \mathcal{B} \quad \text{and} \quad [A] = [A'] \quad \Rightarrow \quad Z(q)_{A\to B} = Z(q)_{A'\to B} \quad \text{for all} \quad B \in \mathcal{B}. \tag{1.24}$$

Actually, we will see in section 2.4 that A and A' cannot be distinguished in cylinder partition functions (1.24) even if A and A' only coincide in the quotient $\widetilde{\mathrm{K}}_0$ defined in (1.17). On the other hand, as opposed to K_0^b one cannot read off the cylinder partition functions directly from the product in $\widetilde{\mathrm{K}}_0$. For example, if $A = 2\mathcal{W}\big(\frac{5}{8}\big)$ and $A' = \mathcal{R}^{(2)}(2, 7)$, the partition functions $Z(q)_{A\to A}$ and $Z(q)_{A'\to A'}$ differ by $2\chi_{\mathcal{W}(0)}(q)$. This difference is visible in K_0^b but not in $\widetilde{\mathrm{K}}_0$. It is therefore not clear to us whether $\widetilde{\mathrm{K}}_0$ has a direct physical interpretation.

It would be interesting to see if the product structure on K_0^b can be described by a Verlinde-like formula; for $\widetilde{\mathrm{K}}_0$, such a formula was obtained in [68].

1.2.4. Boundary conditions and boundary states. Finally, let us comment on the relation between boundary conditions and boundary states. Recall that the boundary states encode the one-point functions of bulk fields on the disc. On the other hand, a boundary condition is in addition specified by the bulk-boundary OPE as well as by the OPE of the boundary fields amongst themselves. This then also specifies how the open string spectra decompose into $\mathcal{W}$-representations.

Given the property (1.24) of cylinder partition functions, it seems likely that the boundary conditions in logarithmic conformal field theories are in general not uniquely characterized by their boundary states. This phenomenon is already visible for the $\mathcal{W}_{1,p}$ models whose boundary theory was analysed in [16, 40]. There, boundary states (and in [40] even the entire boundary condition including OPEs) were constructed for the irreducible representations. The construction of this paper shows that one can find a consistent boundary theory in the sense of B1–B3 also for the other representations (including the indecomposable representations)[7]. The open string spectra of these boundary conditions will still be given by the fusion rules—see (1.21) above and equation (2.21) of [40]—and thus these boundary conditions will be different from the (superpositions of the) irreducible representations that make up the same character. On the other hand, given the analysis of [16, 40] it is clear that there are no additional boundary states, and thus their boundary states must agree.

The same phenomenon is expected to arise for the $\mathcal{W}_{2,3}$ model, although we have not yet constructed the corresponding bulk theory, and thus do not know how many Ishibashi states the theory actually possesses. In fact the above considerations suggest that for the $\mathcal{W}_{2,3}$ model, there are precisely 12 different Ishibashi states since the lattice K_0^b is spanned by 12 characters—see section 2.4.

1.2.5. A boundary theory for other $\mathcal{W}$-symmetric models. While we only consider the $\mathcal{W}_{2,3}$ model in detail in this paper, we believe that much of the structure we have found generalizes to other models (in particular, but not exclusively, to the $\mathcal{W}_{p,q}$ models). The general analysis of section 1.2.1 should be applicable provided that the (interesting) representations of $\mathcal{W}$ form a braided tensor category. One should then be able to find a boundary theory satisfying B1–B3 with boundary labels given by (1.20) and where the open string spectra are of form (1.21). This follows from three additional properties of Rep($\mathcal{W}$) (namely that it is Abelian and has the two properties stated in condition C in section 3.1) together with theorem 3.10. Formulae (1.22)–(1.24) for the cylinder partition functions are also valid in this case, as demonstrated in section 3.5.

2. Representations and fusion rules

After this long summary, we shall now describe our results in more detail. We begin by analysing the fusion rules of the $\mathcal{W}_{2,3}$ model. As we mentioned above, we do not know how to attack this calculation directly, and we shall therefore first revisit the fusion rules of the Virasoro theory.

[7] To connect these to a bulk theory, one obviously still has to construct a consistent bulk-boundary OPE for these additional boundary conditions, but we believe that this is indeed possible.

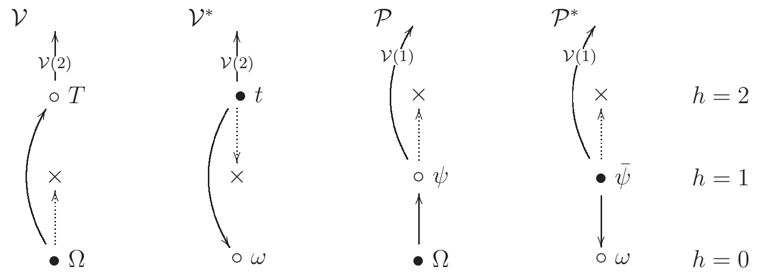
2.1. The Virasoro theory

For the Virasoro theory the relevant vertex operator algebra $\mathcal{V}$ is obtained from the Virasoro Verma module based on Ω, by dividing out $\mathcal{N}_1 = L_{-1}\Omega$, but not $T = L_{-2}\Omega$. The vector T is then a highest weight state, i.e. it is annihilated by L_1 and L_2, but it is not the cyclic vector of $\mathcal{V}$.

Actually, T generates the irreducible representation $\mathcal{V}(2)$ of the Virasoro algebra with highest weight $h = 2$. This representation is the quotient space of the Verma module based on T by two independent null vectors $\mathcal{N}_3$ and $\mathcal{N}_5$ at levels 3 and 5, respectively (see (2.12)). In the Verma module based on Ω, both $\mathcal{N}_3$ and $\mathcal{N}_5$ are actually descendants of $\mathcal{N}_1 = L_{-1}\Omega$ and hence both $\mathcal{N}_3$ and $\mathcal{N}_5$ are set to zero in $\mathcal{V}$. It follows that $\mathcal{V}$ indeed contains $\mathcal{V}(2)$ as a sub-representation. In terms of exact sequences, the structure of the vertex operator algebra is thus

$$0 \longrightarrow \mathcal{V}(2) \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}(0) \longrightarrow 0. \tag{2.1}$$

Alternatively, the structure of the vertex operator algebra is described by the left-most diagram in



(2.2)

Here $\bullet$ denotes the cyclic vector that generates the entire representation, $\circ$ are images of $\bullet$ and $\times$ denotes null vectors (which have been set to zero). The vertex operator algebra $\mathcal{V}$ is then not irreducible, but still indecomposable.

It is easy to see from the above structure that $\mathcal{V}$ is not self-conjugate. This is to say, the two-point correlators involving two states from $\mathcal{V}$ do not lead to a non-degenerate bilinear form. Indeed, it is manifest that

$$\langle \phi(z) T(w) \rangle = 0, \qquad \text{for any} \quad \phi \in \mathcal{V}. \tag{2.3}$$

In fact, the conjugate representation $\mathcal{V}^*$ of $\mathcal{V}$ is generated from a cyclic state t at conformal weight 2. The state t is quasiprimary ($L_1 t = 0$), but it is not highest weight since $L_2 t = \omega$, where ω satisfies $L_n \omega = 0$ for all n. In terms of exact sequences, $\mathcal{V}^*$ is characterized by

$$0 \longrightarrow \mathcal{V}(0) \longrightarrow \mathcal{V}^* \longrightarrow \mathcal{V}(2) \longrightarrow 0, \tag{2.4}$$

and the structure is sketched in the second diagram in (2.2). With this definition, it is then easy to see that

$$\langle T(z) t(w) \rangle = (z - w)^{-2} \langle \Omega(z) \omega(w) \rangle \neq 0. \tag{2.5}$$

This implies that the two-point correlators involving one field from $\mathcal{V}$ and one field from $\mathcal{V}^*$ give rise to a non-degenerate bilinear form.

Both $\mathcal{V}$ and $\mathcal{V}^*$ are obtained from the Verma module based on Ω by taking $\mathcal{N}_2$ to be non-zero. Similarly, we can consider the Virasoro representation where we set $\mathcal{N}_2 = 0$, but not $\psi = L_{-1}\Omega$. This leads to the representations $\mathcal{P}$ and $\mathcal{P}^*$; see (2.2). The representation $\mathcal{P}$

Table 1. The extended Kac table for $c_{2,3} = 0$ showing the values $h_{r,s}$ determined by (1.2).

		s: 1	2	3	4	5	6	7	8	
	1	0	0	$\frac{1}{3}$	1	2	$\frac{10}{3}$	5	7	
	2	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	$\frac{35}{24}$	$\frac{21}{8}$	$\frac{33}{8}$	
r	3	2	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	2	$\cdots$
	4	$\frac{33}{8}$	$\frac{21}{8}$	$\frac{35}{24}$	$\frac{5}{8}$	$\frac{1}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{5}{8}$	
	5	7	5	$\frac{10}{3}$	2	1	$\frac{1}{3}$	0	0	
					$\vdots$					

has a null vector at conformal weight 2, namely

$$\mathcal{N}_2 = \left(L_{-2} - \tfrac{3}{2}L_{-1}^2\right)\Omega. \tag{2.6}$$

This vector is annihilated by L_1, as one can easily verify. (Note that $L_1L_{-2}\Omega = 3L_{-1}\Omega \neq 0$.) Again, the sub-representation generated from ψ is the irreducible Virasoro representation $\mathcal{V}(1)$ with highest weight $h = 1$; its Verma module has independent null vectors at levels 4 and 6, but these are automatically zero in $\mathcal{P}$ since they are descendants of $\mathcal{N}_2$. We can also describe the structure of these representations more formally in terms of exact sequences:

$$0 \longrightarrow \mathcal{V}(1) \longrightarrow \mathcal{P} \longrightarrow \mathcal{V}(0) \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{V}(0) \longrightarrow \mathcal{P}^* \longrightarrow \mathcal{V}(1) \longrightarrow 0. \tag{2.7}$$

The vertex operator algebra $\mathcal{V}$ does not define a rational theory. However, it has a family of 'quasirational' representations [52] that will play an important role later when we enlarge $\mathcal{V}$ to a rational $\mathcal{W}$-algebra. These quasirational representations are labelled by entries in the extended Kac table; see table 1.

2.2. The Virasoro fusion rules

The fusion rules of the vertex operator algebra $\mathcal{V}$ were studied in [20]. The simplest fusion products are those of the irreducible representation $\mathcal{V}(0)$. Since for all n, $L_n\Omega = 0$ in $\mathcal{V}(0)$, the fusion of $\mathcal{V}(0)$ with any state that is in the image of a Virasoro generator, i.e. that can be written as a sum of states of the form $L_n\chi$, vanishes. In particular, this is the case for any state in the irreducible representation $\mathcal{V}(h)$ with $h \neq 0$. On the other hand the product of $\mathcal{V}(0)$ with itself just gives $\mathcal{V}(0)$ again. Thus, we conclude that[8]

$$\mathcal{V}(0) \otimes_{\mathcal{V}} \mathcal{V}(0) = \mathcal{V}(0), \qquad \mathcal{V}(0) \otimes_{\mathcal{V}} \mathcal{V}(h_{r,s}) = 0 \qquad \text{for} \quad h_{r,s} \neq 0. \tag{2.8}$$

The next simplest fusion rules are those that involve the representation $\mathcal{V}(2)$. It was claimed in [20] that $\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2) = \mathcal{V}$ but this is inconsistent with associativity. Indeed, if we assume associativity, then it follows that

$$\mathcal{V}(0) \otimes_{\mathcal{V}} \mathcal{V} = \mathcal{V}(0) \otimes_{\mathcal{V}} (\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2)) = (\mathcal{V}(0) \otimes_{\mathcal{V}} \mathcal{V}(2)) \otimes_{\mathcal{V}} \mathcal{V}(2) = 0 \otimes_{\mathcal{V}} \mathcal{V}(2) = 0, \tag{2.9}$$

but this is not possible since $\mathcal{V}$ is the vertex operator algebra, and hence fusion with $\mathcal{V}$ must always act as the identity, $\mathcal{V}(0) \otimes_{\mathcal{V}} \mathcal{V} = \mathcal{V}(0)$.

[8] Throughout this paper, we shall denote 'fusion' by the symbol $\otimes$. In order to distinguish it from the tensor product over the complex numbers, we shall denote the latter by $\otimes_{\mathbb{C}}$. We shall also reserve $\otimes$ for the $\mathcal{W}$ fusions to be considered below; fusion of $\mathcal{V}$ representations will be denoted by $\otimes_{\mathcal{V}}$.

In order to resolve this issue, we reanalysed the fusion $\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2)$ using the algorithm of [52, 53] (that was also used in [20]). In this approach, the fusion of two representations $\mathcal{H}_1$ and $\mathcal{H}_2$ of the chiral algebra $\mathcal{A}$ is the product space:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 := (\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2)/(\Delta_{z,w} - \tilde{\Delta}_{z,w}), \tag{2.10}$$

where we quotient $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ by the subspace generated by $(\Delta_{z,w}(S_n) - \tilde{\Delta}_{z,w}(S_n))\chi$. Here $\chi \in \mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$, S_n is an arbitrary element of the chiral algebra $\mathcal{A}$ and $\Delta_{z,w}$ and $\tilde{\Delta}_{z,w}$ are the two comultiplication actions of [69]. Furthermore, z and w are the two points in the complex plane where the representations $\mathcal{H}_1$ and $\mathcal{H}_2$ are inserted. The fusion product $\mathcal{H}_1 \otimes \mathcal{H}_2$ then carries an action of the chiral algebra, given by either $\Delta_{z,w}$ or $\tilde{\Delta}_{z,w}$.

In order to unravel the structure of this fusion product, one then considers a family of quotient spaces, the most important of which is the quotient of $\mathcal{H}$ by the states that are in the image of the negative modes,

$$\mathcal{H}^{(0)} := \mathcal{H}/\mathcal{A}_{<0}\mathcal{H}, \tag{2.11}$$

where $\mathcal{A}_{<0}\mathcal{H}$ is the subspace spanned by the states of the form $S_{-n}\chi$ with $n > 0$. If $\mathcal{H}$ is an irreducible highest weight representation, then $\mathcal{H}^{(0)}$ is spanned by the highest weight state. However, one can also determine $\mathcal{H}^{(0)}$ for the case of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Using the above definition of the fusion product (2.10), the quotient space can be calculated algorithmically. Let us illustrate the analysis for the case of $\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2)$. We denote the highest weight vector of $\mathcal{V}(2)$ at conformal weight $h = 2$ by μ. As we have already mentioned before, μ has two independent null vectors, namely

$$\begin{aligned}
\mathcal{N}_3 &= \left(L_{-3} - L_{-2}L_{-1} + \tfrac{1}{6}L_{-1}^3\right)\mu, \\
\mathcal{N}_5 &= \left(L_{-5} - \tfrac{3}{2}L_{-4}L_{-1} - \tfrac{16}{13}L_{-3}L_{-2} + \tfrac{3}{4}L_{-3}L_{-1}^2 + \tfrac{16}{13}L_{-2}^2L_{-1} - \tfrac{15}{26}L_{-2}L_{-1}^3 + \tfrac{9}{208}L_{-1}^5\right)\mu.
\end{aligned} \tag{2.12}$$

For the case of the Virasoro modes, the comultiplications are

$$\begin{aligned}
\Delta_{1,0}(L_0) &= L_{-1} \otimes_{\mathbb{C}} \mathbb{1} + L_0 \otimes_{\mathbb{C}} \mathbb{1} + \mathbb{1} \otimes_{\mathbb{C}} L_0, \\
\Delta_{1,0}(L_{-1}) &= L_{-1} \otimes_{\mathbb{C}} \mathbb{1} + \mathbb{1} \otimes_{\mathbb{C}} L_{-1}, \\
\Delta_{1,0}(L_{-n}) &= \sum_{m=-1}^{\infty} \binom{n+m-1}{m+1} (-1)^{m+1} L_m \otimes_{\mathbb{C}} \mathbb{1} + \mathbb{1} \otimes_{\mathbb{C}} L_{-n}, \qquad n \geqslant 2 \\
\tilde{\Delta}_{0,-1}(L_{-n}) &= L_{-n} \otimes_{\mathbb{C}} \mathbb{1} + \sum_{m=-1}^{\infty} \binom{n+m-1}{m+1} (-1)^{n+1} \mathbb{1} \otimes_{\mathbb{C}} L_m, \qquad n \geqslant 2.
\end{aligned}$$

On the space $(\mathcal{H}_1 \otimes \mathcal{H}_2)^{(0)}$, the action of $\tilde{\Delta}_{0,-1}(L_{-n})$ can also be divided out since $\tilde{\Delta}_{0,-1}(L_{-n})$ only differs by the action of negative modes from $\Delta_{1,0}(L_{-n})$.

First, we use the null vector $\mathcal{N}_3$ at level 3 to conclude that $(\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2))^{(0)}$ must be contained in

$$\operatorname{span}\left\{\left(L_{-1}^n\mu\right) \otimes_{\mathbb{C}} \mu\right\} \supset (\mathcal{V}(2) \otimes \mathcal{V}(2))^{(0)}, \qquad n = 0, 1, 2. \tag{2.13}$$

Using $\mathcal{N}_5$ and the fact that $L_{-1}\mathcal{N}_3$ and $L_{-1}^2\mathcal{N}_3$ are also null in $\mathcal{V}(2)$, we find that we have the relation

$$\left(L_{-1}^2\mu\right) \otimes_{\mathbb{C}} \mu \cong -7(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 8\mu \otimes_{\mathbb{C}} \mu. \tag{2.14}$$

These are all the relations that can be extracted from the null vectors $\mathcal{N}_3$ and $\mathcal{N}_5$, and we therefore conclude that

$$(\mathcal{V}(2) \otimes_{\mathcal{V}} \mathcal{V}(2))^{(0)} = \operatorname{span}\{\mu \otimes_{\mathbb{C}} \mu, (L_{-1}\mu) \otimes_{\mathbb{C}} \mu\}. \tag{2.15}$$

On this space, the L_0 action is then given by

$$\begin{aligned}\Delta_{1,0}(L_0)(\mu\otimes_{\mathbb{C}}\mu) &= (L_{-1}\mu)\otimes_{\mathbb{C}}\mu + 4\mu\otimes_{\mathbb{C}}\mu\\ \Delta_{1,0}(L_0)((L_{-1}\mu)\otimes_{\mathbb{C}}\mu) &= \left(L_{-1}^2\mu\right)\otimes_{\mathbb{C}}\mu + 5(L_{-1}\mu)\otimes_{\mathbb{C}}\mu\\ &\cong -2(L_{-1}\mu)\otimes_{\mathbb{C}}\mu - 8\mu\otimes_{\mathbb{C}}\mu.\end{aligned} \tag{2.16}$$

Thus, we can represent it by the matrix

$$L_0 = \begin{pmatrix}4 & -8\\ 1 & -2\end{pmatrix} \qquad \text{which is conjugate to} \qquad \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}. \tag{2.17}$$

This shows that $\mathcal{V}(2)\otimes_{\mathcal{V}}\mathcal{V}(2)$ contains precisely two vectors (of conformal weights 0 and 2) that are not images under the action of the negative Virasoro modes. In particular, the fusion product is therefore *not* equal to $\mathcal{V}$—it is obvious from the definition that $\mathcal{V}^{(0)} = \mathbb{C}\Omega$. On the other hand, the result is consistent with[9]

$$\mathcal{V}(2)\otimes_{\mathcal{V}}\mathcal{V}(2) = \mathcal{V}^*. \tag{2.18}$$

We have actually checked (2.18) up to level 2, by considering larger quotient spaces as in [53], and our results are perfectly consistent with (2.18). In particular, we have checked that the action of L_2 maps the state at conformal weight 2 to the state at conformal weight 0. Also note that with (2.18) instead of $\mathcal{V}(2)\otimes_{\mathcal{V}}\mathcal{V}(2) = \mathcal{V}$, the problem with (2.9) is resolved; now associativity implies that

$$\mathcal{V}(0)\otimes_{\mathcal{V}}\mathcal{V}^* = 0, \tag{2.19}$$

and this is actually independently correct, since every state in $\mathcal{V}^*$ is in the image of a (possibly positive) Virasoro mode.

We have similarly reanalysed the fusions of $\mathcal{V}(1)$, and instead of the claim of [20] we find

$$\mathcal{V}(1)\otimes_{\mathcal{V}}\mathcal{V}(2) = \mathcal{P}^*, \qquad \mathcal{V}(1)\otimes_{\mathcal{V}}\mathcal{V}(1) = \mathcal{V}^*\oplus\mathcal{V}\left(\tfrac{1}{3}\right). \tag{2.20}$$

On the other hand, we have no reason to believe that there are problems with the other fusion rules of [20], and we have in fact reproduced a number of them independently. We therefore believe that their results are otherwise correct, and we have used a few of them in our analysis of the $\mathcal{W}$ fusion rules below.

It is also worth pointing out that some of the indecomposable representations that appear in the fusions are not just characterized by their highest weight, but also by some additional parameters [5, 53, 70, 71]. In particular, this is the case for the presentations $\mathcal{R}^{(2)}(0,2)_5$ and $\mathcal{R}^{(2)}(0,2)_7$ of [20], for which the relevant parameter is called β_2 and is listed in table 2 of that paper. Incidentally, the two values for β_2 agree precisely with what was already determined in [12] using slightly different methods, although the interpretation is now different; in [12] it was thought that this implied that only some subsector of representations could consistently exist in a given theory. In the present context (see also [7]), in particular in connection with our boundary analysis below, we see that both representations appear in the same theory, but never together in an open string spectrum $\mathcal{H}_{A\to B}$ for indecomposable $\mathcal{W}$-representations A, B.

2.3. $\mathcal{W}$-representations

Up to now, we have only considered the Virasoro theory. In order to make this theory rational, we have to extend it by adjoining three states at conformal weight $h = 15$. We shall denote the resulting vertex operator algebra $\mathcal{W}$. As in the Virasoro case discussed before, $\mathcal{W}$ is again not irreducible and its structure is similar to that of $\mathcal{V}$.

[9] This has also been independently observed by Jørgen Rasmussen. We thank him for communicating this to us.

The $\mathcal{W}$ theory is expected to have only finitely many irreducible representations. They can be expressed in terms of infinite sums of quasirational Virasoro representations [18, section 3.5]. The irreducible representations are characterized by the eigenvalue of L_0 on the ground state, and we shall denote them by $\mathcal{W}(h)$. They contain in particular the irreducible Virasoro representation $\mathcal{V}(h)$. In our case, 13 irreducible $\mathcal{W}$-representations appear, and their conformal weights are listed in the Kac table (1.4). In addition, we have the $\mathcal{W}$-analogues of the $\mathcal{V}$-representations $\mathcal{V}^*$, $\mathcal{P}$ and $\mathcal{P}^*$. We shall denote them by $\mathcal{W}^*$, $\mathcal{Q}$ and $\mathcal{Q}^*$, respectively. The exact sequences characterizing $\mathcal{W}$ and $\mathcal{W}^*$ have been given in (1.7) and (1.8) respectively. For $\mathcal{Q}$ and $\mathcal{Q}^*$, we have in analogy with (2.7)

$$0 \longrightarrow \mathcal{W}(1) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{W}(0) \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{W}(0) \longrightarrow \mathcal{Q}^* \longrightarrow \mathcal{W}(1) \longrightarrow 0. \tag{2.21}$$

Further logarithmic representations appear in the various fusion products; they were listed in (1.5). The notation is inspired from [20] (but here denotes $\mathcal{W}$-representations, and not $\mathcal{V}$-representations as in [20]).

2.3.1. Lifting the Virasoro fusion to $\mathcal{W}$-fusion. Unfortunately the high weight of the additional $\mathcal{W}$-fields makes it difficult to determine the full chiral algebra explicitly and thus to calculate the fusion directly, using the methods from above. However, we can infer the fusion rules of at least certain $\mathcal{W}$-representations from the corresponding Virasoro fusions using induced representations. These are the $\mathcal{W}$-representations that are of the form

$$\mathcal{H}^{\mathcal{W}} = \mathcal{W} \otimes_{\mathcal{V}} \mathcal{H}^{\mathcal{V}}, \tag{2.22}$$

where $\mathcal{H}^{\mathcal{V}}$ is a Virasoro representation, and $\mathcal{V}$ acts on $\mathcal{W}$ by restricting the $\mathcal{W}$-action to the Virasoro algebra; the fusion is with respect to the Virasoro algebra. The fusion (with respect to the vertex operator algebra $\mathcal{W}$) of two such representations is then

$$\mathcal{H}_1^{\mathcal{W}} \otimes \mathcal{H}_2^{\mathcal{W}} = \big(\mathcal{W} \otimes_{\mathcal{V}} \mathcal{H}_1^{\mathcal{V}}\big) \otimes \big(\mathcal{W} \otimes_{\mathcal{V}} \mathcal{H}_2^{\mathcal{V}}\big) \cong \mathcal{W} \otimes_{\mathcal{V}} \big(\mathcal{H}_1^{\mathcal{V}} \otimes_{\mathcal{V}} \mathcal{H}_2^{\mathcal{V}}\big). \tag{2.23}$$

This allows us to calculate a certain number of $\mathcal{W}$ fusion products, based on our knowledge of the $\mathcal{V}$ fusion rules. Combining these results with associativity, we have managed to determine all $\mathcal{W}$ fusion rules of all the representations we have mentioned above. Our analysis reproduces the results of [6, 21], but it goes beyond their analysis since we can also determine the fusion rules of the irreducible representations $\mathcal{W}(1)$, $\mathcal{W}(2)$, $\mathcal{W}(5)$ and $\mathcal{W}(7)$, as well as of the indecomposable representations $\mathcal{W}^*$ and $\mathcal{Q}^*$—the fusion rules involving $\mathcal{W}$ and $\mathcal{Q}$ were recently conjectured in [21]. For example, we find that the fusion of $\mathcal{W}(0)$ is trivial except for

$$\mathcal{W}(0) \otimes \mathcal{W}(0) = \mathcal{W}(0). \tag{2.24}$$

The fusion with $\mathcal{W}$ acts as the identity on every representation. Furthermore, we find

$$\begin{array}{ll}
\mathcal{W}(2) \otimes \mathcal{W}(2) = \mathcal{W}^* & \mathcal{W}^* \otimes \mathcal{W}^* = \mathcal{W}^* \\
\mathcal{W}(2) \otimes \mathcal{W}^* = \mathcal{W}^* & \mathcal{W}^* \otimes \mathcal{Q} = \mathcal{Q}^* \\
\mathcal{W}(2) \otimes \mathcal{Q} = \mathcal{W}(1) & \mathcal{W}^* \otimes \mathcal{Q}^* = \mathcal{Q}^* \\
\mathcal{W}(2) \otimes \mathcal{Q}^* = \mathcal{Q}^* & \mathcal{W}^* \otimes \mathcal{W}(1) = \mathcal{Q}^* \\
\mathcal{W}(2) \otimes \mathcal{W}(1) = \mathcal{Q}^*, &
\end{array} \tag{2.25}$$

and

$$\begin{array}{ll}
\mathcal{Q} \otimes \mathcal{Q} = \mathcal{W} \oplus \mathcal{W}\big(\tfrac{1}{3}\big) & \mathcal{Q}^* \otimes \mathcal{Q}^* = \mathcal{W}^* \oplus \mathcal{W}\big(\tfrac{1}{3}\big) \\
\mathcal{Q} \otimes \mathcal{Q}^* = \mathcal{W}^* \oplus \mathcal{W}\big(\tfrac{1}{3}\big) & \mathcal{Q}^* \otimes \mathcal{W}(1) = \mathcal{W}^* \oplus \mathcal{W}\big(\tfrac{1}{3}\big) \\
\mathcal{Q} \otimes \mathcal{W}(1) = \mathcal{W}\big(\tfrac{1}{3}\big) \oplus \mathcal{W}(2) & \mathcal{W}(1) \otimes \mathcal{W}(1) = \mathcal{W}^* \oplus \mathcal{W}\big(\tfrac{1}{3}\big).
\end{array} \tag{2.26}$$

The complete list of fusion products is given in appendix A.4.

2.4. Multiplication on the Grothendieck group

Finally, we turn to the question to which extent the fusion of representations induces a product on the space of characters. More formally, we want to study the Grothendieck group $K_0 = K_0(\mathrm{Rep}(\mathcal{W}_{2,3}))$ of the tensor category $\mathrm{Rep}(\mathcal{W}_{2,3})$. We already saw in section 1.1.3 that it is inconsistent to define the product (1.15) on all of K_0. However, as we will recall in section 3.5, if a representation $\mathcal{M}$ has a dual representation then we get a well-defined map $K_0 \to K_0$ given by

$$[\mathcal{R}] \mapsto [\mathcal{M} \otimes \mathcal{R}]. \tag{2.27}$$

In words this means that if two representations $\mathcal{R}$ and $\mathcal{R}'$ have the same character, then so have $\mathcal{M} \otimes \mathcal{R}$ and $\mathcal{M} \otimes \mathcal{R}'$. We denote by K_0^r the subgroup of K_0 generated by $[\mathcal{R}]$ for all $\mathcal{R}$ which have a dual representation[10]. It is slightly larger than the subgroup K_0^b introduced in (1.23).

Note that if a representation has a dual representation, the dual need not be the conjugate representation. For example one finds that the $\mathcal{W}$-algebra always has $\mathcal{W}$ as a dual (see lemma 3.7), even though in the $\mathcal{W}_{2,3}$ model the conjugate representation $\mathcal{W}^*$ is not isomorphic to $\mathcal{W}$. Similarly, since $\mathcal{W}$ appears as a direct summand in the fusion $\mathcal{Q} \otimes \mathcal{Q}$, we expect $\mathcal{Q}$ to be self-dual. However, the representation $\mathcal{Q}$ is again not isomorphic to its conjugate $\mathcal{Q}^*$. In any case, we have already found two elements of K_0^r, namely

$$[\mathcal{W}], \quad [\mathcal{Q}] \in K_0^r. \tag{2.28}$$

As pointed out in section 1.1.1, we expect the representations from (1.4) and (1.5) which are not in grey boxes to have duals. These include

$$[\mathcal{W}(h)] \in K_0^r \qquad \text{for} \quad h \in \left\{\tfrac{1}{3}, \tfrac{10}{3}, \tfrac{5}{8}, \tfrac{33}{8}, \tfrac{1}{8}, \tfrac{21}{8}, \tfrac{-1}{24}, \tfrac{35}{24}\right\}, \tag{2.29}$$

and

$$[\mathcal{R}^{(2)}(0,2)_7], \quad [\mathcal{R}^{(2)}(2,7)], \quad [\mathcal{R}^{(2)}(1,5)] \in K_0^r. \tag{2.30}$$

The other representations in (1.5) that are not in grey boxes define elements in K_0 that can be expressed as integer linear combinations of the generators (2.28)–(2.30). Furthermore, by comparing characters, it is not difficult to show that the 13 elements of K_0^r given in (2.28)–(2.30) are linearly independent.

As a consistency check of the claim that the representations in (2.28)–(2.30) have duals, we have verified that the map (2.27) is indeed independent of the choice of representative for $[\mathcal{R}]$, provided we choose $\mathcal{M}$ from (2.28)–(2.30). We have also verified that the remaining representations cannot have duals. For example, the characters in appendix A.1 show that $[\mathcal{R}^{(2)}(0,2)_7] = [\mathcal{W}(0) \oplus 2\mathcal{W}(2) \oplus 2\mathcal{W}(7)]$, but if we take $\mathcal{M} = \mathcal{W}^*, \mathcal{Q}^*, \mathcal{W}(0), \mathcal{W}(1), \mathcal{W}(2), \mathcal{W}(5)$ or $\mathcal{W}(7)$, the

$$[\mathcal{M} \otimes \mathcal{R}^{(2)}(0,2)_7] \neq [\mathcal{M} \otimes (\mathcal{W}(0) \oplus 2\mathcal{W}(2) \oplus 2\mathcal{W}(7))], \tag{2.31}$$

and thus none of these $\mathcal{M}$ can have duals. We will prove in lemma 3.7 that if two representations $\mathcal{R}$ and $\mathcal{R}'$ have duals, then so does their fusion product $\mathcal{R} \otimes \mathcal{R}'$. This shows that the multiplication $[\mathcal{R}] \cdot [\mathcal{R}'] := [\mathcal{R} \otimes \mathcal{R}']$ provides a well-defined associative product on K_0^r. This can also be verified explicitly, using the table of fusion products in appendix A.4.

It would be natural to work in a basis consisting of the irreducible representations, but this is not quite possible. To see this, note that $[\mathcal{R}^{(2)}(0,2)_7] - [\mathcal{R}^{(2)}(2,7)] = [\mathcal{W}(0)]$, so that[11] $[\mathcal{W}(0)] \in K_0^r$. Because $[\mathcal{W}] = [\mathcal{W}(0)] + [\mathcal{W}(2)]$, then also $[\mathcal{W}(2)] \in K_0^r$. In the

[10] Here r stands for 'rigid'; see definition 3.6.

[11] Note that the fact that $[\mathcal{W}(0)] \in K_0^r$ does not imply that $\mathcal{W}(0)$ has a dual representation; as we have seen in (2.31), it does not.

same way, $[\mathcal{Q}] \in \mathrm{K}_0^r$ implies $[\mathcal{W}(1)] \in \mathrm{K}_0^r$. Finally, $[\mathcal{R}^{(2)}(2,7)] - 2[\mathcal{W}(2)] = 2[\mathcal{W}(7)]$ and $[\mathcal{R}^{(2)}(1,5)] - 2[\mathcal{W}(1)] = 2[\mathcal{W}(5)]$. Altogether, we see that

$$\mathrm{K}_0^r = \mathrm{span}_{\mathbb{Z}}\big([\mathcal{W}(h)]\big|h = 0, 2, 1, \tfrac{1}{3}, \tfrac{10}{3}, \tfrac{5}{8}, \tfrac{33}{8}, \tfrac{1}{8}, \tfrac{21}{8}, \tfrac{-1}{24}, \tfrac{35}{24}\big) \oplus 2\mathbb{Z}[\mathcal{W}(7)] \oplus 2\mathbb{Z}[\mathcal{W}(5)]. \tag{2.32}$$

In particular, $[\mathcal{W}(5)]$ and $[\mathcal{W}(7)]$ are *not in* K_0^r, but only $2[\mathcal{W}(5)]$ and $2[\mathcal{W}(7)]$.

For completeness, let us also work out some of the structure constants in the basis given by (2.32). Note that by construction, these structure constants will be integers, but they need not be non-negative. Indeed, the product in this basis cannot just be calculated by taking $[\mathcal{R}]\cdot[\mathcal{R}'] := [\mathcal{R}\otimes\mathcal{R}']$—this formula is only true if both $\mathcal{R}$ and $\mathcal{R}'$ have duals. Thus in order to calculate the structure constants in the basis (2.32), we have to rewrite the generators in terms of (2.28)–(2.30) and then use the product formulae for these. For example, $[\mathcal{W}(0)]\cdot[\mathcal{W}(0)]$ is *not* given by $[\mathcal{W}(0)\otimes\mathcal{W}(0)] = [\mathcal{W}(0)]$, because $\mathcal{W}(0)$ does not have a dual. Instead, we have to write $[\mathcal{W}(0)] = [\mathcal{R}^{(2)}(0,2)_7] - [\mathcal{R}^{(2)}(2,7)]$ and compute

$$\begin{aligned}[\mathcal{W}(0)]\cdot[\mathcal{W}(0)] &= [\mathcal{R}^{(2)}(0,2)_7 \otimes \mathcal{R}^{(2)}(0,2)_7] - [\mathcal{R}^{(2)}(2,7)\otimes\mathcal{R}^{(2)}(0,2)_7] \\ &\quad - [\mathcal{R}^{(2)}(0,2)_7\otimes\mathcal{R}^{(2)}(2,7)] + [\mathcal{R}^{(2)}(2,7)\otimes\mathcal{R}^{(2)}(2,7)] = 0.\end{aligned} \tag{2.33}$$

This also explains why the problem encountered in (1.16) when trying to define a multiplication on K_0 does not occur for K_0^r: while it is true that $[\mathcal{W}^*] = [\mathcal{W}(0)] + [\mathcal{W}(2)] \in \mathrm{K}_0^r$, the product $[\mathcal{W}^*]\cdot[\mathcal{W}(0)]$ is not given by $[\mathcal{W}^*\otimes\mathcal{W}(0)]$, because neither $\mathcal{W}^*$ nor $\mathcal{W}(0)$ have duals. Instead, we have to express $[\mathcal{W}^*] = [\mathcal{W}(0)] + [\mathcal{W}(2)]$ in terms of representations which do have duals. For example, we can use $[\mathcal{W}^*] = [\mathcal{W}]$ and express $[\mathcal{W}(0)]$ as above. The result is $[\mathcal{W}^*]\cdot[\mathcal{W}(0)] = [\mathcal{W}(0)]$. Similarly, the action of $[\mathcal{W}(0)]$ in the basis (2.32) is fixed by

$$[\mathcal{W}(0)]\cdot[\mathcal{W}(h)] = \begin{cases}[\mathcal{W}(0)] & \text{if} \quad h = 1,2\\ -[\mathcal{W}(0)] & \text{if} \quad h = 5,7\\ 0 & \text{else.}\end{cases} \tag{2.34}$$

In particular, this implies that $\mathbb{Z}[\mathcal{W}(0)]$ is an ideal in K_0^r. We can thus consider the quotient space $\widetilde{\mathrm{K}}_0$, see (1.17), with the quotient map $\pi : \mathrm{K}_0^r \to \widetilde{\mathrm{K}}_0$ given by $\pi([\mathcal{R}]) = [\mathcal{R}] + \mathbb{Z}[\mathcal{W}(0)]$. Because of the factors of 2 in (2.32) the map π is not surjective, but one can check that there is a (unique) multiplication on $\widetilde{\mathrm{K}}_0$ such that π is a ring homomorphism (this is not obvious because the factors of 2 in (2.32) might lead to non-integer structure constants for $\widetilde{\mathrm{K}}_0$). We have verified that the product on $\widetilde{\mathrm{K}}_0$ obtained in this way agrees with the one in [18, section 6.3] that was constructed using quantum groups (see appendix A.2 for how to translate the notations).

The representations (1.20) which actually correspond to boundary conditions generate the subgroup K_0^b of K_0^r. Compared to K_0^r, the group K_0^b misses the generators $[\mathcal{W}]$ and $[\mathcal{Q}]$, but we have to add $[\mathcal{R}^{(2)}(2,5)]$ which could previously be expressed as $[\mathcal{R}^{(2)}(1,5)] + 2[\mathcal{W}] - 2[\mathcal{Q}]$. Writing the basis in a similar fashion as (2.32) gives

$$\begin{aligned}\mathrm{K}_0^b = \mathrm{span}_{\mathbb{Z}}&\big([\mathcal{W}(h)]\big|h = 0, \tfrac{1}{3}, \tfrac{10}{3}, \tfrac{5}{8}, \tfrac{33}{8}, \tfrac{1}{8}, \tfrac{21}{8}, \tfrac{-1}{24}, \tfrac{35}{24}\big)\\ &\oplus 2\mathbb{Z}([\mathcal{W}(2)] + [\mathcal{W}(7)]) \oplus 2\mathbb{Z}([\mathcal{W}(1)] + [\mathcal{W}(5)]) \oplus 2\mathbb{Z}([\mathcal{W}(2)] + [\mathcal{W}(5)]).\end{aligned} \tag{2.35}$$

In particular, the lattice K_0^b has only 12 basis vectors. We will prove in theorem 3.9 that also K_0^b is closed under multiplication. To see this explicitly, we observe that (2.35) is the kernel of the map $[\mathcal{R}] \mapsto [\mathcal{W}(0)]\cdot[\mathcal{R}]$, which goes from K_0^r to itself. This description of K_0^b implies that any two representations $\mathcal{R}$ and $\mathcal{R}'$ with the property that $[\mathcal{R}] - [\mathcal{R}'] = n[\mathcal{W}(0)]$ for some n will have the same cylinder partition function relative to any $[\mathcal{S}] \in \mathrm{K}_0^b$. Indeed, we have in K_0^b

$$[\mathcal{R}]\cdot[\mathcal{S}^*] = [\mathcal{R}']\cdot[\mathcal{S}^*] + ([\mathcal{R}]-[\mathcal{R}'])\cdot[\mathcal{S}^*] = [\mathcal{R}']\cdot[\mathcal{S}^*] + n[\mathcal{W}(0)]\cdot[\mathcal{S}^*] = [\mathcal{R}']\cdot[\mathcal{S}^*], \tag{2.36}$$

where we have used that $n[\mathcal{W}(0)]\cdot[\mathcal{S}^*] = 0$ in K_0^b.

3. Internal Homs and associativity

With this detailed understanding of the fusion rules, we are now in a position to construct a boundary theory for the $\mathcal{W}_{2,3}$ model. We shall switch gears and formulate our construction in a more categorical fashion. First, we shall explain informally why category theory is the appropriate language (see sections 3.1 and 3.2). Then we shall introduce the relevant notions that will be important to us, in particular that of an internal Hom (see section 3.3) and that of dual objects (see section 3.4). There, we also establish the result announced in section 1, namely that for the list of boundary labels (1.20) we can find a boundary theory satisfying conditions B1–B3.

3.1. Tensor categories

From the results in section 2, we see that it is reasonable to assume that the $\mathcal{W}_{2,3}$-representations we consider form a tensor category. This is a non-trivial assumption (see the discussion in section 1.1.4), as a tensor category contains quite a lot of structure.

A tensor category is a tuple $\mathcal{C} \equiv (\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is the tensor product bifunctor, $\mathbf{1} \in \mathcal{C}$ is the tensor unit, $\alpha_{U,V,W} : U\otimes(V\otimes W) \xrightarrow{\sim} (U\otimes V)\otimes W$ is the associator, $\lambda_U : \mathbf{1}\otimes U \to U$ is the left unit isomorphism and $\rho_U : U\otimes\mathbf{1} \to U$ is the right unit isomorphism. These data are subject to conditions; in particular α satisfies the pentagon axiom and λ, ρ, α obey the triangle axiom. For more details on tensor categories, the reader could consult [57, 72].

For a tensor category Rep($\mathcal{W}$) arising as the representations of a suitable vertex operator algebra $\mathcal{W}$, we expect three additional features. First of all, Rep($\mathcal{W}$) should be Abelian, so that in particular we can speak about kernels and quotients. Second, to each representation $\mathcal{R}$ we can assign a conjugate representation[12] $\mathcal{R}^*$ such that $\mathcal{R}^{**} \cong \mathcal{R}$. In fact, we have a contravariant functor $(-)^*$ from Rep($\mathcal{W}$) to itself whose square is naturally equivalent to the identity functor [61, notation 2.36]. Finally, for two representations $\mathcal{R}$ and $\mathcal{S}$ there is an isomorphism between the spaces of intertwiners $\mathrm{Hom}(\mathcal{R},\mathcal{S}^*)$ and $\mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{W}^*)$. To see this, note that $\mathrm{Hom}(\mathcal{R},\mathcal{S}^*)$ is by definition isomorphic to the space of conformal two-point blocks on the complex plane with insertions of $\mathcal{R}$ at 0 and $\mathcal{S}$ at ∞ with standard local coordinates, and $\mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{W}^*)$ is isomorphic to the space of conformal three-point blocks with insertions of $\mathcal{R}$ at 0, $\mathcal{S}$ at a point $z \in \mathbb{C}^\times$ and $\mathcal{W}$ at ∞, all treated as 'in-going' punctures. Since an insertion of the vertex operator algebra itself does not affect the dimension of the space of conformal blocks, these two spaces are isomorphic[13].

The last two properties motivate the following condition.

[12] In the vertex operator algebra literature, this representation is usually referred to as the 'contragredient' representation.

[13] This also follows from the study of logarithmic intertwiners [61]. Indeed, $\mathrm{Hom}(\mathcal{R},\mathcal{S}^*) \cong \mathrm{Hom}(\mathcal{R}\otimes\mathcal{W},\mathcal{S}^*)$. The latter space is by construction the space of intertwiners from $\mathcal{R}\times\mathcal{W}$ to $\mathcal{S}^*$. By [61, proposition 3.46], this space is naturally isomorphic to the space of intertwiners from $\mathcal{R}\times\mathcal{S}^{**}$ to $\mathcal{W}^*$. Using that $\mathcal{S}^{**}\cong\mathcal{S}$, this shows that we have a natural isomorphism $\mathrm{Hom}(\mathcal{R},\mathcal{S}^*) \cong \mathrm{Hom}(\mathcal{R}\otimes\mathcal{S},\mathcal{W}^*)$. We thank Yi-Zhi Huang for a discussion of this point.

Condition C. The tensor category $\mathcal{C}$ is equipped with a contravariant involutive functor $(-)^* : \mathcal{C} \to \mathcal{C}$ and isomorphisms $\pi_{U,V} : \mathrm{Hom}(U, V^*) \to \mathrm{Hom}(U \otimes V, \mathbf{1}^*)$ which are natural in U and V.

We denote the natural isomorphism from the identity functor on $\mathcal{C}$ to the square of $(-)^*$ by

$$\delta_U : U \to U^{**}. \tag{3.1}$$

3.2. Associative and non-degenerate boundary operator product expansion

Next, we want to describe in more detail conditions B1–B3 of section 1.2.1 that any consistent boundary theory should satisfy. Some of these conditions are nothing but the usual requirement that the sewing constraints [66] must be satisfied. We shall only spell them out in some detail to make the categorical conditions below look less mysterious. Let us start with the OPE. Consider two boundary fields $\psi \in \mathcal{H}_{A\to B}$ and $\psi' \in \mathcal{H}_{B\to C}$,

$$\xrightarrow{\;A\;} \underset{\psi(0)}{\times} \xrightarrow{\;B\;} \underset{\psi'(x)}{\times} \xrightarrow{\;C\;} \tag{3.2}$$

The line in this picture is the boundary of the upper half-plane, i.e. the real axis with standard orientation. The OPE of two such fields is described by a bilinear map[14]

$$V_{C,B,A}(-, x) : \mathcal{H}_{B\to C} \times \mathcal{H}_{A\to B} \to \mathcal{H}_{A\to C} \tag{3.3}$$

taking (ψ', ψ) to $V_{C,B,A}(\psi', x)\psi$. The map $V_{C,B,A}(-, x)$ has to be compatible with the $\mathcal{W}$-symmetry in the sense of intertwining operators (see e.g. [61, section 3]). Condition B1 demands the OPE to be associative. Consider three boundary fields as follows:

$$\xrightarrow{\;A\;} \underset{\psi_1(0)}{\times} \xrightarrow{\;B\;} \underset{\psi_2(y)}{\times} \xrightarrow{\;C\;} \underset{\psi_3(x)}{\times} \xrightarrow{\;D\;} \tag{3.4}$$

Associativity means that it does not matter if we first take the OPE of ψ_2 with ψ_1 or that of ψ_3 with ψ_2. Written out in terms of the bilinear maps V, this condition reads as

$$V_{D,C,A}(\psi_3, x)V_{C,B,A}(\psi_2, y)\psi_1 = V_{D,B,A}(V_{D,C,B}(\psi_3, x-y)\psi_2, y)\psi_1. \tag{3.5}$$

Condition B2 states that the bilinear pairing

$$(\psi', \psi) \longmapsto \langle \varepsilon_A(V_{A,B,A}(\psi', x)\psi), \Omega\rangle \tag{3.6}$$

on $\mathcal{H}_{B\to A} \times \mathcal{H}_{A\to B}$ is non-degenerate. Recall that ε_A is an intertwiner from $\mathcal{H}_{A\to A}$ to $\mathcal{W}^*$, so that one can evaluate the result on the vacuum vector $\Omega \in \mathcal{W}$. Finally, B3 requires in particular that $\eta_A(\Omega)$ is the identity field on A, i.e. for $\psi \in \mathcal{H}_{A\to B}$,

$$V_{B,B,A}(\eta_B(\Omega), x)\psi = \psi \qquad \text{and} \qquad \lim_{x\to 0} V_{B,A,A}(\psi, x)\eta_A(\Omega) = \psi. \tag{3.7}$$

We now reformulate (3.3), (3.5)–(3.7) in a way which makes sense in an arbitrary tensor category satisfying condition C. There are morphisms

$$m_{C,B,A} \in \mathrm{Hom}(\mathcal{H}_{B\to C} \otimes \mathcal{H}_{A\to B}, \mathcal{H}_{A\to C}), \qquad \eta_A : \mathbf{1} \to \mathcal{H}_{A\to A}, \qquad \varepsilon_A : \mathcal{H}_{A\to A} \to \mathbf{1}^*, \tag{3.8}$$

[14] This is not quite right as the $\mathcal{W}$-representations are defined as direct sums of generalized L_0-eigenspaces, and the map $V_{C,B,A}$ in general has contributions in an infinite number of generalized L_0-eigenspaces. Instead one should use formal power series, including formal logarithms. We refer to [61, section 3] for more details.

such that

$$\begin{array}{ll}
\text{(associativity)} & m_{D,C,A}\circ(\mathrm{id}_{\mathcal{H}_{C\to D}}\otimes m_{C,B,A}) \\
& \quad = m_{D,B,A}\circ(m_{D,C,B}\otimes \mathrm{id}_{\mathcal{H}_{A\to B}})\circ\alpha_{\mathcal{H}_{C\to D},\mathcal{H}_{B\to C},\mathcal{H}_{A\to B}} \\
\text{(unit property)} & m_{B,B,A}\circ(\eta_B\otimes \mathrm{id}_{\mathcal{H}_{A\to B}}) = \lambda_{\mathcal{H}_{A\to B}}, \quad m_{B,A,A}\circ(\mathrm{id}_{\mathcal{H}_{A\to B}}\otimes\eta_A) = \rho_{\mathcal{H}_{A\to B}} \\
\text{(non-degeneracy)} & \pi^{-1}_{\mathcal{H}_{B\to A},\mathcal{H}_{A\to B}}(\varepsilon_A\circ m_{A,B,A}) \text{ is an isomorphism.}
\end{array} \tag{3.9}$$

Here α, λ and ρ are defined in section 3.1 and π is the isomorphism of condition C. The three conditions above amount to B1, B2 and half of B3. To account for all of B3, we need to require in addition that η_A is injective.

3.3. Internal Homs

Usually the most difficult condition in constructing a consistent boundary theory is B1, the associativity of the OPE. In the following, we shall describe a general construction—the internal Hom—which solves this condition automatically. In this subsection, we shall give a brief overview of some properties of internal Homs; for more information, the reader could consult e.g. [73, section 9.3].

Definition 3.3. *Let $\mathcal{C}$ be a tensor category. Given two objects $A, B \in \mathcal{C}$, an internal Hom from A to B is an object $[A, B] \in \mathcal{C}$ together with a natural isomorphism $\phi^{(A,B)} : \mathrm{Hom}(-\otimes A, B) \to \mathrm{Hom}(-, [A, B])$.*

Naturality of $\phi^{(A,B)}$ is equivalent to the statement that for all $X, Y \in \mathcal{C}$ and all $g : Y \to X$, $t : X \otimes A \to B$,

$$\phi_X^{(A,B)}(t)\circ g = \phi_Y^{(A,B)}(t\circ(g\otimes \mathrm{id}_A)). \tag{3.10}$$

An internal Hom need not exist, but if it does it is unique up to unique isomorphism. For suppose that $[A, B]$ and $[A, B]'$ are internal Homs from A to B with natural isomorphisms $\phi^{(A,B)}$ and $\phi^{(A,B)'}$. Then there exists a unique isomorphism $f : [A, B] \to [A, B]'$ such that the following diagram commutes for all $U \in \mathcal{C}$:

$$\begin{tikzcd}
& \mathrm{Hom}(U\otimes A, B) \arrow[dl, "\phi_U^{(A,B)}"'] \arrow[dr, "\phi_U^{(A,B)'}"] & \\
\mathrm{Hom}(U, [A,B]) \arrow[rr, "f\circ(-)"] & & \mathrm{Hom}(U, [A,B]')
\end{tikzcd} \tag{3.11}$$

The morphism f is obtained by taking $U = [A, B]$ and using $\phi_{[A,B]}^{(A,B)^{-1}}$ and $\phi_{[A,B]}^{(A,B)'}$ to transport $\mathrm{id}_{[A,B]}$ to $\mathrm{Hom}([A, B], [A, B]')$.

As an example of an internal Hom, consider the category of finite-dimensional complex vector spaces. If A, B are two such vector spaces, then $[A, B] = B \otimes_{\mathbb{C}} A^*$, i.e. the space of linear maps from A to B. Indeed, if $f : U \otimes A \to B$ is a homomorphism, then $\phi_U^{(A,B)}(f)$ is a homomorphism from $U \to B \otimes_{\mathbb{C}} A^*$. Evaluated on $u \in U$, $\left[\phi_U^{(A,B)}(f)\right](u)$ is an element of $B \otimes_{\mathbb{C}} A^*$, and thus a homomorphism from $A \to B$, which agrees with $f(u, -)$.

Internal Homs also provide a different way of stating the second part of condition C. It is equivalent to $[V, \mathbf{1}^*] = V^*$.

For Hom spaces of a category, there is an associative composition. For internal Hom spaces there is an analogous concept, which we review now (see e.g. [73, proposition 9.3.13] or [74, section 3.2]). Define the morphisms (*evaluation*, *multiplication* and *unit* for internal Homs)

$$\begin{aligned} &ev_{A,B} : [A,B]\otimes A \to B, \qquad ev_{A,B} = \phi^{(A,B)-1}_{[A,B]}(\mathrm{id}_{[A,B]}),\\ &m_{C,B,A} : [B,C]\otimes[A,B] \to [A,C],\\ &m_{C,B,A} = \phi^{(A,C)}_{[B,C]\otimes[A,B]}((ev_{B,C}\circ(\mathrm{id}_{[B,C]}\otimes ev_{A,B}))\circ\alpha^{-1}_{[B,C],[A,B],A}),\\ &\eta_A : \mathbf{1}\to[A,A], \qquad \eta_A = \phi^{(A,A)}_{\mathbf{1}}(\lambda_A). \end{aligned} \tag{3.12}$$

Theorem 3.2. *The composition of internal Homs is associative, i.e. on* $[C,D]\otimes([B,C]\otimes[A,B])$ *we have*

$$m_{D,C,A}\circ(\mathrm{id}_{[C,D]}\otimes m_{C,B,A}) = m_{D,B,A}\circ(m_{D,C,B}\otimes\mathrm{id}_{[A,B]})\circ\alpha_{[C,D],[B,C],[A,B]},$$

and it has η *as unit, i.e. on* $\mathbf{1}\otimes[A,B]$ *and* $[A,B]\otimes\mathbf{1}$ *we have*

$$m_{B,B,A}\circ(\eta_B\otimes\mathrm{id}_{[A,B]}) = \lambda_{[A,B]}, \qquad m_{B,A,A}\circ(\mathrm{id}_{[A,B]}\otimes\eta_A) = \rho_{[A,B]}.$$

The proof is by straightforward calculation. We spell it out for completeness in appendix B.1. The following theorem shows that internal Homs exist if the tensor category satisfies condition C.

Theorem 3.3. *Let* $\mathcal{C}$ *be a tensor category satisfying condition* C*. Then* $[A,B] = (A\otimes B^*)^*$ *is an internal Hom from* A *to* B.

Proof. Consider the sequence of isomorphisms

$$\begin{aligned} &\mathrm{Hom}(U\otimes A,B) \xrightarrow{\delta_B\circ(-)} \mathrm{Hom}(U\otimes A,B^{**}) \xrightarrow{\pi_{U\otimes A,B^*}} \mathrm{Hom}((U\otimes A)\otimes B^*,\mathbf{1}^*)\\ &\xrightarrow{(-)\circ\alpha_{U,A,B^*}} \mathrm{Hom}(U\otimes(A\otimes B^*),\mathbf{1}^*) \xrightarrow{\pi^{-1}_{U,A\otimes B^*}} \mathrm{Hom}(U,(A\otimes B^*)^*). \end{aligned} \tag{3.13}$$

The above isomorphisms are all natural in U, and as a consequence so is $\phi^{(A,B)}_U$: $\mathrm{Hom}(U\otimes A,B)\to\mathrm{Hom}(U,(A\otimes B^*)^*)$,

$$\phi^{(A,B)}_U(f) = \pi^{-1}_{U,A\otimes B^*}(\pi_{U\otimes A,B^*}(\delta_B\circ f)\circ\alpha_{U,A,B^*}). \tag{3.14}$$

This shows that $(A\otimes B^*)^*$ is an internal Hom from A to B. □

For the reasons stated in section 3.1, we think it likely that $\mathrm{Rep}(\mathcal{W}_{2,3})$ is an Abelian tensor category satisfying condition C. It therefore has internal Homs. This in turn allows us to find a boundary theory which satisfies condition B1 by setting $\mathcal{H}_{A\to B} = [A,B]$ and choosing the morphisms defined in (3.12). Associativity is guaranteed by theorem 3.2. However, in general, η_A need not be injective nor need there exist a non-degenerate two-point correlator. We will address these two problems in the following section with the help of dual objects.

3.4. Dual objects

The notion of a dual object in a tensor category is a generalization of the properties of the dual of a finite dimensional vector space. For a finite dimensional vector space V over $\mathbb{C}$ (say), there is a linear map $d_V : V^*\otimes_{\mathbb{C}} V\to\mathbb{C}$ given by evaluation: $d_V(\varphi\otimes_{\mathbb{C}} v) = \varphi(v)$. Conversely, if we fix a basis v_i of V and denote the dual basis by v_i^* we obtain a linear map $b_V : \mathbb{C}\to V\otimes_{\mathbb{C}} V^*$ as $\lambda\mapsto\lambda\sum_i v_i\otimes_{\mathbb{C}} v_i^*$. One checks that these maps have the properties

$$(\mathrm{id}_V\otimes_{\mathbb{C}} d_V)\circ(b_V\otimes_{\mathbb{C}}\mathrm{id}_V) = \mathrm{id}_V \quad\text{and}\quad (d_V\otimes_{\mathbb{C}}\mathrm{id}_{V^*})\circ(\mathrm{id}_{V^*}\otimes_{\mathbb{C}} b_V) = \mathrm{id}_{V^*}. \tag{3.15}$$

This notion is generalized to arbitrary tensor categories as follows (see e.g. [57, definition 2.1.1] or [73, definition 9.3.1]).

Definition 3.4. *Let $\mathcal{C}$ be a tensor category. A right dual of an object U is an object $U^\vee \in \mathcal{C}$ together with morphisms $b_U : \mathbf{1} \to U \otimes U^\vee$ and $d_U : U^\vee \otimes U \to \mathbf{1}$ such that*

$$\rho_U \circ (\mathrm{id}_U \otimes d_U) \circ \alpha^{-1}_{U,U^\vee,U} \circ (b_U \otimes \mathrm{id}_U) \circ \lambda_U^{-1} = \mathrm{id}_U \quad \text{and}$$
$$\lambda_{U^\vee} \circ (d_U \otimes \mathrm{id}_{U^\vee}) \circ \alpha_{U^\vee,U,U^\vee} \circ (\mathrm{id}_{U^\vee} \otimes b_U) \circ \rho^{-1}_{U^\vee} = \mathrm{id}_{U^\vee}.$$

Just as did condition C, right duals guarantee the existence of internal Homs and so allow us to solve condition B1 for a boundary theory.

Lemma 3.5. *Let $\mathcal{C}$ be a tensor category. If $U \in \mathcal{C}$ has a right dual, then for all $V \in \mathcal{C}$ we can choose $[U, V] = V \otimes U^\vee$.*

Proof. For $f : A \otimes U \to V$, define $\phi_A^{(U,V)}(f) : A \to V \otimes U^\vee$ as

$$\phi_A^{(U,V)}(f) = (f \otimes \mathrm{id}_{U^\vee}) \circ \alpha_{A,U,U^\vee} \circ (\mathrm{id}_A \otimes b_U) \circ \rho_A^{-1}. \tag{3.16}$$

The properties of b_U and d_U can be used to check that the map $\tilde{\phi}_A^{(U,V)}(g) : A \otimes U \to V$ defined for $g : A \to V \otimes U^\vee$ by

$$\tilde{\phi}_A^{(U,V)}(g) = \rho_V \circ (\mathrm{id}_V \otimes d_U) \circ \alpha^{-1}_{V,U^\vee,U} \circ (g \otimes \mathrm{id}_U) \tag{3.17}$$

is a left and right inverse to ϕ_A. Thus, ϕ_A is an isomorphism. Naturality follows by writing out both sides of (3.10) and using naturality of ρ and α and functoriality of the tensor product. □

Substituting the explicit expressions (3.16) and (3.17) into (3.12) gives the following result for the multiplication m and unit morphisms η (we do not spell out the unit isomorphisms and associators of the tensor category $\mathcal{C}$):

$$\begin{aligned} m_{C,B,A} &= \mathrm{id}_C \otimes d_B \otimes \mathrm{id}_A : (C \otimes B^\vee) \otimes (B \otimes A^\vee) \to C \otimes A^\vee, \\ \eta_A &= b_A : \mathbf{1} \to A \otimes A^\vee. \end{aligned} \tag{3.18}$$

Similar to right duals, one defines the *left dual* of an object U as an object ${}^\vee U$ together with morphisms $\tilde{b}_U : \mathbf{1} \to {}^\vee U \otimes U$ and $\tilde{d}_U : U \otimes {}^\vee U \to \mathbf{1}$ satisfying conditions analogous to those for right duals; see [57, section 2.1]. The representation categories of (suitable) vertex operator algebras are not only tensor categories, but they also have a braiding and a twist. This additional structure ensures that every left dual is also a right dual and vice versa [56, section 7]. We take this as a motivation to not single out right duals and instead treat both on the same footing.

Definition 3.6. *Let $\mathcal{C}$ be a tensor category. The category $\mathcal{C}^r$ of rigid objects in $\mathcal{C}$ is the full sub-category consisting of all objects $U \in \mathcal{C}$ that have a right and a left dual.*

Not every object in a tensor category need to have a right and/or left dual. However, the tensor unit $\mathbf{1}$ always has itself as a right and a left dual, and the objects which have right and left duals form a full tensor sub-category.

Lemma 3.7. $\mathbf{1} \in \mathcal{C}^r$*, and for $U, V \in \mathcal{C}^r$ also $U \otimes V \in \mathcal{C}^r$.*

Proof. It is easy to check that one can choose $\mathbf{1}^\vee = {}^\vee\mathbf{1} = \mathbf{1}$ with $b_\mathbf{1} = \tilde{b}_\mathbf{1} = \lambda_\mathbf{1}^{-1}$ and $d_\mathbf{1} = \tilde{d}_\mathbf{1} = \lambda_\mathbf{1}$. For $U \otimes V$ we set $(U \otimes V)^\vee := V^\vee \otimes U^\vee$ and, omitting the associator and unit isomorphisms, $b_{U\otimes V} = (\mathrm{id}_U \otimes b_V \otimes \mathrm{id}_{U^\vee}) \circ b_U$ and $d_{U\otimes V} = d_V \circ (\mathrm{id}_{V^\vee} \otimes d_U \otimes \mathrm{id}_V)$. The verification of the properties in definition 3.4 is straightforward. That ${}^\vee(U \otimes V) := {}^\vee V \otimes {}^\vee U$ is a left dual can be checked in the same way. □

The category $\mathcal{C}^r$ allows us to solve condition B1 in the construction of a boundary theory, but does still not guarantee B2 and B3. For example in Rep($\mathcal{W}_{2,3}$) the $\mathcal{W}$ algebra is the tensor unit, and so is its own left and right duals. But as already pointed out in section 1.2.2, $\mathcal{W} \ncong \mathcal{W}^*$ and so $\mathcal{W}$ does not allow for a non-degenerate two-point correlator. Instead, we will consider the following sub-category of $\mathcal{C}^r$.

Definition 3.8. *Let $\mathcal{C}$ be a tensor category satisfying condition C. Then $\mathcal{C}^b$ (where b stands for 'boundary') denotes the sub-category of $\mathcal{C}$ consisting of all objects U for which U^* is both a right dual and a left dual of U and for which both $b_U : \mathbf{1} \to U \otimes U^*$ and $\tilde{b}_U : \mathbf{1} \to U^* \otimes U$ are injective.*

The injectivity requirement will guarantee the injectivity of the unit morphisms in (3.18). Note that even if $\mathcal{C}$ is Abelian, $\mathcal{C}^b$ is not. For example, it does not contain the zero object $\mathbf{0}$ as $b_{\mathbf{0}} = 0$ is not injective. The uniqueness of internal Homs, together with theorem 3.3 and lemma 3.5, implies that

$$(A \otimes B^*)^* \cong B \otimes A^* \qquad \text{for} \quad A, B \in \mathcal{C}^b. \tag{3.19}$$

The following theorem shows that $\mathcal{C}^b$ is closed under taking conjugates and tensor products. It will be proved in appendix B.2.

Theorem 3.9. *Let $\mathcal{C}$ be a tensor category satisfying condition* C.

(i) If $U \in \mathcal{C}^b$, then also $U^* \in \mathcal{C}^b$.
(ii) If $U, V \in \mathcal{C}^b$, then also $U \otimes V \in \mathcal{C}^b$.

Note that (ii) does not imply that $\mathcal{C}^b$ is a tensor category, because in general $\mathbf{1} \notin \mathcal{C}^b$. As we have just seen, the $\mathcal{W}_{2,3}$ model provides an example for this. On the category $\mathcal{C}^b$, we can define a boundary theory satisfying B1–B3. We choose $\mathcal{B}$ to consist of the objects of $\mathcal{C}^b$. For the open string state spaces we again take $\mathcal{H}_{A\to B} = [A, B]$, but now we choose the internal Hom defined in lemma 3.5, i.e. $[A, B] = B \otimes A^*$. Multiplication and unit morphisms are defined by (3.18). Property B1—associativity of the multiplication—holds by theorem 3.2, but it can also be easily verified directly. For the one-point correlation function, we choose

$$\varepsilon_A = \pi_{A,A^*}(\delta_A) : [A, A] \to \mathbf{1}^*, \tag{3.20}$$

where $\delta_A : A \to A^{**}$ was defined in (3.1). Properties B2 and B3 are established in the next theorem, to be proved in appendix B.2.

Theorem 3.10. *Let $\mathcal{C}$ be an Abelian tensor category satisfying condition* C *and let $A, B \in \mathcal{C}^b$. Then $[A, B] = B \otimes A^*$ and*

(i) $B \otimes A^*$ is non-zero,
(ii) the morphism $\eta_A : \mathbf{1} \to [A, A]$ is injective,
(iii) the pairing $\varepsilon_A \circ m_{A,B,A} : [B, A] \otimes [A, B] \to \mathbf{1}^*$ is non-degenerate.

Altogether we see that, provided Rep($\mathcal{W}_{2,3}$) is an Abelian braided tensor category satisfying condition C, we can define a boundary theory satisfying B1–B3 on the set of boundary labels given in (1.20) (in a braided tensor category with twist, b_A is injective iff $\tilde{b}_A$ is injective). We believe that the representations in (1.4) and (1.5) which are not in grey boxes are in Rep$(\mathcal{W}_{2,3})^b$. We verify that b_A are injective in appendix B.3.

3.5. Subgroups of the Grothendieck group

Let $\mathcal{C}$ be an Abelian tensor category. The Grothendieck group $\mathrm{K}_0(\mathcal{C})$ is defined as the free Abelian group generated by isomorphism classes of objects in $\mathcal{C}$, divided by the subgroup generated by the elements $[U]+[W]-[V]$ for each exact sequence $0 \to U \to V \to W \to 0$ in $\mathcal{C}$; see e.g. [57, definition 2.1.9]. Note that this definition implies in particular that $[U \oplus V] = [U]+[V]$.

If an object $A \in \mathcal{C}$ has the property that for each exact sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ also $0 \to A \otimes U \xrightarrow{\mathrm{id}_A \otimes f} A \otimes V \xrightarrow{\mathrm{id}_A \otimes g} A \otimes W \to 0$ is exact, we say that the functor $A \otimes (-)$ is exact. This is not always the case, as we have seen explicitly in section 1.1.2.

However, if $A \otimes (-)$ is exact, then we get a well-defined map $[U] \mapsto [A \otimes U]$ on $K_0(\mathcal{C})$. It is proved in [57, proposition 2.1.8] that if A has both a left and a right dual, then both $A \otimes (-)$ and $(-) \otimes A$ are exact. This motivates the definition

$$\mathrm{K}_0^r(\mathcal{C}) = (\text{the subgroup of } K_0(\mathcal{C}) \text{ generated by } [U] \text{ for all } U \in \mathcal{C}^r\ ). \tag{3.21}$$

By lemma 3.7, the assignment $([U],[V]) \mapsto [U \otimes V]$ gives a well-defined map $\mathrm{K}_0^r(\mathcal{C}) \times \mathrm{K}_0^r(\mathcal{C}) \to \mathrm{K}_0^r(\mathcal{C})$. Because the tensor product is associative, this map defines an associative product on $\mathrm{K}_0^r(\mathcal{C})$ with unit element $[\mathbf{1}]$. Thus even if the tensor product does not induce a product on $K_0(\mathcal{C})$, we always have a unital ring structure on the Abelian subgroup $\mathrm{K}_0^r(\mathcal{C}) \subset \mathrm{K}_0(\mathcal{C})$.

In the context of boundary conformal field theory the representation category is expected to be an Abelian tensor category satisfying property C, and we have seen that we can associate a boundary theory with the category $\mathcal{C}^b$. We can then define a corresponding subgroup of the Grothendieck group:

$$\mathrm{K}_0^b(\mathcal{C}) = (\text{the subgroup of } K_0(\mathcal{C}) \text{ generated by } [U] \text{ for all } U \in \mathcal{C}^b\ ). \tag{3.22}$$

By definition $\mathrm{K}_0^b(\mathcal{C}) \subset \mathrm{K}_0^r(\mathcal{C})$, and by theorem 3.9 the product on $\mathrm{K}_0^r(\mathcal{C})$ restricts to a product on $\mathrm{K}_0^b(\mathcal{C})$. Because $\mathcal{C}^b$ does not necessarily have a unit, neither does $\mathrm{K}_0^b(\mathcal{C})$.

4. Conclusions and outlook

In this paper, we have studied the $\mathcal{W}_{2,3}$ triplet model in some detail. In particular, we have determined the fusion rules of the theory, i.e. we have determined the fusion rules of all representations that appear in successive fusions of the irreducible representations. (The complete list of fusion rules is given in appendix A.4.) We have also studied some of the unusual properties of these representations and their fusions. For example, there is a subtle difference between conjugate and dual representations (see section 1.1.1), and the Grothendieck group K_0 that is generated by the characters of the 13 irreducible representations of the $\mathcal{W}_{2,3}$ model does not admit a straightforward product (see section 2.4).

The second main result concerns a boundary theory for the $\mathcal{W}_{2,3}$ model which is analogous to the Cardy case in non-logarithmic rational conformal field theory. We have identified the subset $\mathcal{B}$ of representations to which we can assign consistent boundary conditions. The resulting boundary conditions have boundary fields whose OPEs are associative (this is guaranteed by the internal Hom construction; see theorem 3.2). In addition, the boundary two-point correlators are non-degenerate (theorem 3.10) and the spectrum of boundary fields between any two such boundary conditions is non-empty (see theorem 3.9).

The representations in $\mathcal{B}$ are characterized by the property that the conjugate agrees with the dual representation and that the intertwiner $b_{\mathcal{R}}$ that is needed for duality is an injection; see (1.20). If we restrict the Grothendieck group K_0 to $\mathcal{B}$—this defines the group K_0^b that is generated by 12 independent characters (see (2.35))—then the fusion rules lead to a well-defined product which characterizes the cylinder partition functions between these boundary conditions.

Our analysis of the boundary theory did not rely on the details of the corresponding bulk theory and, indeed, the idea of the approach is to try and reconstruct the bulk theory starting from our boundary analysis. However, it is *a priori* not clear whether this will be possible, and thus the construction of the corresponding bulk theory is the main open problem that remains for the $\mathcal{W}_{2,3}$ model. A good starting point might be the observation in [75, 16] that, for certain supergroup WZW models and for the $\mathcal{W}_{1,p}$ triplet models, the space of bulk states $\mathcal{H}_{\text{bulk}}$ is a quotient of

$$\bigoplus_i P_i \otimes_{\mathbb{C}} \bar{P}_i^*, \tag{4.1}$$

where the sum runs over the indecomposable projective representations and the bar refers to right movers. Furthermore, at least in these examples, the character of the quotient was given by

$$Z(q) = \sum_i \chi_{P_i}(q) \cdot \chi_{\bar{U}_i^*}(\bar{q}), \tag{4.2}$$

where U_i is the irreducible representation of which P_i is the projective cover.

In analogy with the $\mathcal{W}_{1,p}$ models, it seems likely to us that the irreducible $\mathcal{W}_{2,3}$-representation $\mathcal{W}\big(\frac{-1}{24}\big)$ is projective. If $\mathcal{P}$ is projective and $\mathcal{R}$ has a dual, then $\mathcal{R}^\vee \otimes \mathcal{P}$ is also projective. Therefore, if $\mathcal{W}\big(\frac{-1}{24}\big)$ is projective, the following 12 $\mathcal{W}_{2,3}$-representations have to be projective as well

$$\begin{aligned}
&\mathcal{W}\big(\tfrac{-1}{24}\big), \quad \mathcal{W}\big(\tfrac{35}{24}\big),\\
&\mathcal{R}^{(2)}\big(\tfrac{1}{3}, \tfrac{1}{3}\big), \quad \mathcal{R}^{(2)}\big(\tfrac{1}{3}, \tfrac{10}{3}\big), \quad \mathcal{R}^{(2)}\big(\tfrac{5}{8}, \tfrac{5}{8}\big), \quad \mathcal{R}^{(2)}\big(\tfrac{5}{8}, \tfrac{21}{8}\big), \quad \mathcal{R}^{(2)}\big(\tfrac{1}{8}, \tfrac{1}{8}\big),\\
&\mathcal{R}^{(2)}\big(\tfrac{1}{8}, \tfrac{33}{8}\big), \quad \mathcal{R}^{(3)}(0, 0, 1, 1), \quad \mathcal{R}^{(3)}(0, 0, 2, 2), \quad \mathcal{R}^{(3)}(0, 1, 2, 5), \quad \mathcal{R}^{(3)}(0, 1, 2, 7).
\end{aligned} \tag{4.3}$$

The fusion product of such a projective representation with any other representation from (1.4) or (1.5) produces a direct sum of representations in (4.3), and we therefore think that these are all indecomposable projective representations. In fact, by comparison with the embedding diagrams in appendix A.3 one finds that these are the projective covers of the irreducible representations:

$$\begin{aligned}
&\mathcal{W}\big(\tfrac{-1}{24}\big), \quad \mathcal{W}\big(\tfrac{35}{24}\big),\\
&\mathcal{W}\big(\tfrac{1}{3}\big), \quad \mathcal{W}\big(\tfrac{10}{3}\big), \quad \mathcal{W}\big(\tfrac{5}{8}\big), \quad \mathcal{W}\big(\tfrac{21}{8}\big), \quad \mathcal{W}\big(\tfrac{1}{8}\big), \quad \mathcal{W}\big(\tfrac{33}{8}\big),\\
&\mathcal{W}(1), \quad \mathcal{W}(2), \quad \mathcal{W}(5), \quad \mathcal{W}(7),
\end{aligned} \tag{4.4}$$

in this order. We do not know if $\mathcal{W}(0)$ has a projective cover, but if it has, it is not one of the representations we consider in (1.4) and (1.5) (see appendix A.3). The characters of the representations in (4.3) agree, up to overall factors, with the characters of projective representations proposed in [18, section 5.2.1], and the representations themselves agree with the list proposed in [19, section 3.6]. We find by inspection that ansatz (4.2) by itself is not modular invariant, but the following slight modification is:

$$\begin{aligned}
Z_{\mathcal{W}_{2,3}}(q) &= (n\chi_{\mathcal{W}(0)}(q) + 2\chi_{\mathcal{W}(1)}(q) + 2\chi_{\mathcal{W}(2)}(q) + 2\chi_{\mathcal{W}(5)}(q) + 2\chi_{\mathcal{W}(7)}(q)) \cdot \chi_{\mathcal{W}(0)}(\bar{q})\\
&\quad + \sum_i \chi_{P_i}(q) \cdot \chi_{\bar{U}_i^*}(\bar{q})\\
&= (q\bar{q})^{-1/24} + n + 2(q\bar{q})^{1/8} + 2(q\bar{q})^{1/3} + (q + \bar{q}) \cdot (q\bar{q})^{-1/24} + 2(q + \bar{q}) + \cdots,
\end{aligned} \tag{4.5}$$

where the sum runs over the 12 projectives P_i in (4.3) with the corresponding irreducibles U_i given in (4.4). The integer n is not constrained by modular invariance, but since there is at least one vector of conformal weight 0 we have $n \geqslant 1$. Even if it is not apparent from the way it is written, expression (4.5) is left/right symmetric. Furthermore, it agrees with the modular invariant combination of characters given in [18, section 5.3] (for $n = 1$ and up to an overall factor of 4). The extra term in (4.5) with respect to (4.2) could indicate that for the $\mathcal{W}_{2,3}$ model, the quotient of $\bigoplus_i P_i \otimes_{\mathbb{C}} \bar{P}_i^*$ needed to obtain $\mathcal{H}_{\text{bulk}}$ is more complicated. It is amusing to note that for $n = 1$, we can write (4.5) as

$$Z_{\mathcal{W}_{2,3}}(q) = \sum_i \dim(\text{Hom}(P_i, P_i))^{-2} \cdot |\chi_{P_i}(q)|^2, \tag{4.6}$$

where the sum extends over the 12 projective representations. Incidentally, this formula also works for the $\mathcal{W}_{1,p}$ models as well as the non-logarithmic rational theories since for these theories (4.6) and (4.2) agree. This suggests that (4.6) could be more generally true.

Finally, let us remark that the size of the Grothendieck group K_0^b suggests that one needs 12 Ishibashi states to construct the boundary states for the boundary conditions in $\mathcal{B}$. This coincides with the number of projective representations in (4.3), and thus one could guess that one needs one Ishibashi state from each summand in (4.1). Incidentally, this is precisely what happened for the $\mathcal{W}_{1,p}$ models [16].

It would be very interesting to study the $\mathcal{W}_{2,3}$ bulk theory in more detail and to see whether these expectations are indeed borne out. We also expect that much of the structure we have discovered for the $\mathcal{W}_{2,3}$ model holds more generally for the $\mathcal{W}_{p,q}$ models. We hope to return to these questions in the near future.

Acknowledgments

We would like to thank Michael Flohr, Jürgen Fuchs, Yi-Zhi Huang, Jørgen Rasmussen, David Ridout and Christoph Schweigert for helpful discussions and useful comments on a draft of this paper. IR thanks ETH Zürich and the Centre for Theoretical Studies for hospitality during the final stages of this work. The research of MRG and SW is supported by the Swiss National Science Foundation. IR is partially supported by the EPSRC First Grant EP/E005047/1 and the STFC Rolling Grant ST/G000395/1.

Appendix A. More on representations and fusion rules

A.1. Characters

Let us first list the characters of all the irreducible representations; these were given in [18, section 5.1]. We use the formulation in [6, section 3.2] where the characters of the indecomposable $\mathcal{R}^{(\cdot)}(\cdots)$ representations in (1.5) can also be found:

$$\begin{aligned}
\chi_{\mathcal{W}(0)} &= 1 \\
\chi_{\mathcal{W}(1)} &= \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 (q^{(12k-7)^2/24} - q^{(12k+1)^2/24}) \\
&= q(1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + \cdots) \\
\chi_{\mathcal{W}(2)} &= \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 (q^{(12k-5)^2/24} - q^{(12k-1)^2/24}) \\
&= q^2(1 + q + 2q^2 + 2q^3 + 4q^4 + 4q^5 + \cdots)
\end{aligned}$$

$$\chi_{\mathcal{W}(5)} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1)\left(q^{(12k-1)^2/24} - q^{(12k+7)^2/24}\right)$$
$$= q^5(2 + 2q + 4q^2 + 6q^3 + 10q^4 + 14q^5 + \cdots)$$
$$\chi_{\mathcal{W}(7)} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1)\left(q^{(12k+1)^2/24} - q^{(12k+5)^2/24}\right)$$
$$= q^7(2 + 2q + 4q^2 + 6q^3 + 10q^4 + 12q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{1}{3})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{3(4k-3)^2/8} = q^{1/3}(1 + q + 2q^2 + 2q^3 + 4q^4 + 5q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{10}{3})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k q^{3(4k-1)^2/8} = q^{10/3}(2 + 2q + 4q^2 + 6q^3 + 10q^4 + 14q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{1}{8})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{(6k-5)^2/6} = q^{1/8}(1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{5}{8})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{(6k-4)^2/6} = q^{5/8}(1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{21}{8})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k q^{(6k-2)^2/6} = q^{21/8}(2 + 2q + 4q^2 + 6q^3 + 10q^4 + 14q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{33}{8})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k q^{(6k-1)^2/6} = q^{33/8}(2 + 2q + 4q^2 + 6q^3 + 8q^4 + 12q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{-1}{24})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{(6k-6)^2/6} = q^{-1/24}(1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots)$$
$$\chi_{\mathcal{W}(\frac{35}{24})} = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k q^{(6k-3)^2/6} = q^{35/24}(2 + 2q + 4q^2 + 6q^3 + 10q^4 + 14q^5 + \cdots).$$

In terms of the irreducible representations, the rank 1 representations have the characters

$$\chi_{\mathcal{W}} = \chi_{\mathcal{W}^*} = 1 + \chi_{\mathcal{W}(2)}, \qquad \chi_{\mathcal{Q}} = \chi_{\mathcal{Q}^*} = 1 + \chi_{\mathcal{W}(1)},$$

while the characters of the rank 2 representations are

$$\begin{aligned}
\chi_{\mathcal{R}^{(2)}(\frac{1}{3},\frac{1}{3})} &= \chi_{\mathcal{R}^{(2)}(\frac{1}{3},\frac{10}{3})} = 2\chi_{\mathcal{W}(\frac{1}{3})} + 2\chi_{\mathcal{W}(\frac{10}{3})} \\
\chi_{\mathcal{R}^{(2)}(\frac{1}{8},\frac{1}{8})} &= \chi_{\mathcal{R}^{(2)}(\frac{1}{8},\frac{33}{8})} = 2\chi_{\mathcal{W}(\frac{1}{8})} + 2\chi_{\mathcal{W}(\frac{33}{8})} \\
\chi_{\mathcal{R}^{(2)}(\frac{5}{8},\frac{5}{8})} &= \chi_{\mathcal{R}^{(2)}(\frac{5}{8},\frac{21}{8})} = 2\chi_{\mathcal{W}(\frac{5}{8})} + 2\chi_{\mathcal{W}(\frac{21}{8})} \\
\chi_{\mathcal{R}^{(2)}(0,2)_7} &= 1 + \chi_{\mathcal{R}^{(2)}(2,7)} = 1 + 2\chi_{\mathcal{W}(2)} + 2\chi_{\mathcal{W}(7)} \\
\chi_{\mathcal{R}^{(2)}(0,1)_5} &= 1 + \chi_{\mathcal{R}^{(2)}(1,5)} = 1 + 2\chi_{\mathcal{W}(1)} + 2\chi_{\mathcal{W}(5)} \\
\chi_{\mathcal{R}^{(2)}(0,1)_7} &= 1 + \chi_{\mathcal{R}^{(2)}(1,7)} = 1 + 2\chi_{\mathcal{W}(1)} + 2\chi_{\mathcal{W}(7)} \\
\chi_{\mathcal{R}^{(2)}(0,2)_5} &= 1 + \chi_{\mathcal{R}^{(2)}(2,5)} = 1 + 2\chi_{\mathcal{W}(2)} + 2\chi_{\mathcal{W}(5)}.
\end{aligned}$$

Finally, all rank 3 representations have the same character

$$\chi_{\mathcal{R}^{(3)}(0,k,\ell,m)} = 2\chi_{\mathcal{W}(0)} + 4\chi_{\mathcal{W}(1)} + 4\chi_{\mathcal{W}(2)} + 4\chi_{\mathcal{W}(5)} + 4\chi_{\mathcal{W}(7)}. \tag{A.1}$$

A.2. Dictionary of the notation in other works

The notation in [6, 21]. It is straightforward to identify the irreducible representations by comparing the conformal weight of the ground state. We can then successively identify the indecomposable representations by comparing the fusions of these representations. As a non-trivial consistency check, we have also compared the embedding diagrams[15] of [20, Figures 2–5] with the embedding diagrams of [6, 21]; see in particular the diagram [6, equation (3.34)], relations [6, equations (3.35), (3.41)] and diagram [21, equation (4.9)].

Our notation	Notation in [6, 21]	Our notation	Notation in [6, 21]
$\mathcal{W}$	$(1,1)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(0,1)_5$	$(\mathcal{R}^{1,0}_{2,2})_{\mathcal{W}}$
$\mathcal{Q}$	$(1,2)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(1,5)$	$(\mathcal{R}^{1,0}_{4,2})_{\mathcal{W}}$
$\mathcal{W}(\frac{1}{3})$	$(1,3)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(0,2)_7$	$(\mathcal{R}^{1,0}_{2,1})_{\mathcal{W}}$
$\mathcal{W}(\frac{10}{3})$	$(1,6)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(2,7)$	$(\mathcal{R}^{1,0}_{4,1})_{\mathcal{W}}$
$\mathcal{W}(\frac{5}{8})$	$(2,1)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{1}{3},\frac{1}{3})$	$(\mathcal{R}^{1,0}_{2,3})_{\mathcal{W}}$
$\mathcal{W}(\frac{33}{8})$	$(4,1)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{1}{3},\frac{10}{3})$	$(\mathcal{R}^{1,0}_{2,6})_{\mathcal{W}} = (\mathcal{R}^{1,0}_{4,3})_{\mathcal{W}}$
$\mathcal{W}(\frac{1}{8})$	$(2,2)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{5}{8},\frac{5}{8})$	$(\mathcal{R}^{0,2}_{2,3})_{\mathcal{W}}$
$\mathcal{W}(\frac{21}{8})$	$(4,2)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{1}{8},\frac{33}{8})$	$(\mathcal{R}^{0,2}_{2,6})_{\mathcal{W}}$
$\mathcal{W}(\frac{-1}{24})$	$(2,3)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{1}{8},\frac{1}{8})$	$(\mathcal{R}^{0,1}_{2,3})_{\mathcal{W}}$
$\mathcal{W}(\frac{35}{24})$	$(2,6)_{\mathcal{W}} = (4,3)_{\mathcal{W}}$	$\mathcal{R}^{(2)}(\frac{5}{8},\frac{21}{8})$	$(\mathcal{R}^{0,1}_{2,6})_{\mathcal{W}}$
$\mathcal{R}^{(2)}(0,1)_7$	$(\mathcal{R}^{0,1}_{1,3})_{\mathcal{W}}$	$\mathcal{R}^{(3)}(0,0,1,1)$	$(\mathcal{R}^{1,1}_{2,3})_{\mathcal{W}}$
$\mathcal{R}^{(2)}(2,5)$	$(\mathcal{R}^{0,1}_{1,6})_{\mathcal{W}}$	$\mathcal{R}^{(3)}(0,1,2,5)$	$(\mathcal{R}^{1,1}_{2,6})_{\mathcal{W}} = (\mathcal{R}^{1,1}_{4,3})_{\mathcal{W}}$
$\mathcal{R}^{(2)}(0,2)_5$	$(\mathcal{R}^{0,2}_{1,3})_{\mathcal{W}}$	$\mathcal{R}^{(3)}(0,0,2,2)$	$(\mathcal{R}^{1,2}_{2,3})_{\mathcal{W}}$
$\mathcal{R}^{(2)}(1,7)$	$(\mathcal{R}^{0,2}_{1,6})_{\mathcal{W}}$	$\mathcal{R}^{(3)}(0,1,2,7)$	$(\mathcal{R}^{1,2}_{2,6})_{\mathcal{W}} = (\mathcal{R}^{1,2}_{4,3})_{\mathcal{W}}$

The representations $\mathcal{W}(0)$, $\mathcal{W}(1)$, $\mathcal{W}(2)$, $\mathcal{W}(5)$, $\mathcal{W}(7)$, $\mathcal{W}^*$, $\mathcal{Q}^*$ do not appear in [6, 21]. The identifications in the above table are those in [6, equations (3.1), (3.3)].

The notation in [18]. The identification can be made by comparing (1.4) to [18, table 1] and the sequences (1.7) and (2.21) to [18, section 3.4].

Our notation	Notation in [18]
$\mathcal{W}$	$\mathcal{K}^+_{1,1}$
$\mathcal{Q}$	$\mathcal{K}^+_{1,2}$
$\mathcal{W}(0)$	$\mathcal{X}_{1,1}$
$\mathcal{W}(2)$	$\mathcal{X}^+_{1,1}$
$\mathcal{W}(7)$	$\mathcal{K}^-_{1,1} = \mathcal{X}^-_{1,1}$
$\mathcal{W}(1)$	$\mathcal{X}^+_{1,2}$
$\mathcal{W}(5)$	$\mathcal{K}^-_{1,2} = \mathcal{X}^-_{1,2}$

Our notation	Notation in [18]
$\mathcal{W}(\frac{1}{3})$	$\mathcal{K}^+_{1,3} = \mathcal{X}^+_{1,3}$
$\mathcal{W}(\frac{10}{3})$	$\mathcal{K}^-_{1,3} = \mathcal{X}^-_{1,3}$
$\mathcal{W}(\frac{5}{8})$	$\mathcal{K}^+_{2,1} = \mathcal{X}^+_{2,1}$
$\mathcal{W}(\frac{33}{8})$	$\mathcal{K}^-_{2,1} = \mathcal{X}^-_{2,1}$
$\mathcal{W}(\frac{1}{8})$	$\mathcal{K}^+_{2,2} = \mathcal{X}^+_{2,2}$
$\mathcal{W}(\frac{21}{8})$	$\mathcal{K}^-_{2,2} = \mathcal{X}^-_{2,2}$
$\mathcal{W}(\frac{-1}{24})$	$\mathcal{K}^+_{2,3} = \mathcal{X}^+_{2,3}$
$\mathcal{W}(\frac{35}{24})$	$\mathcal{K}^-_{2,3} = \mathcal{X}^-_{2,3}$

The representations $\mathcal{W}^*$, $\mathcal{Q}^*$ and those of the form $\mathcal{R}^{(\cdot)}(\cdots)$ are not considered in [18].

A.3. Embedding structure of the $\mathcal{W}$-representations

For the convenience of the reader, we transcribe the embedding diagrams of [6, (3.34)] in our notation.

[15] Note that the embedding diagrams of [20] describe the Virasoro action, while those of [6, 21] refer to the $\mathcal{W}$ action.

Rank 2 representations. The rank 2 representations are indecomposable combinations of the irreducible representations. For $\ell > h > 0$, they are given in [6] as

$$\mathcal{R}^{(2)}(0,h)_\ell : \quad \begin{tikzcd} \mathcal{W}(\ell) \arrow[d] & & \mathcal{W}(\ell) \arrow[dll] \\ \mathcal{W}(h) & & \mathcal{W}(h) \arrow[ll] \arrow[u] \arrow[ull] \arrow[dl] \\ & \mathcal{W}(0) \arrow[ul] & \end{tikzcd} \tag{A.2}$$

while for h fractional, the diagrams of [6] are

$$\mathcal{R}^{(2)}(h,h) : \quad \begin{tikzcd} \mathcal{W}(h+n) \arrow[d] & & \mathcal{W}(h+n) \arrow[dll] \\ \mathcal{W}(h) & & \mathcal{W}(h) \arrow[ll] \arrow[u] \arrow[ull] \end{tikzcd} \tag{A.3}$$

where $n = 2, 3, 4$ for $h = \frac{5}{8}, \frac{1}{3}, \frac{1}{8}$, respectively. Finally,

$$\mathcal{R}^{(2)}(h,h+n) : \quad \begin{tikzcd} \mathcal{W}(h+n) & & \mathcal{W}(h+n) \arrow[ll] \arrow[dll] \arrow[d] \\ \mathcal{W}(h) \arrow[u] & & \mathcal{W}(h) \arrow[ull] \end{tikzcd} \tag{A.4}$$

where h and $h+n$ are as in the previous case, and additionally $h = 1, 2$ and $h+n = 5, 7$ (for all four combinations).

Rank 3 representations. Similarly, the rank 3 representations are indecomposable combinations of the rank 2 representations. For each representation, two equivalent embedding diagrams are given in [6].

$$\mathcal{R}^{(3)}(0,0,1,1) : \quad \begin{tikzcd} \mathcal{R}^{(2)}(2,7) \arrow[d] & & \mathcal{R}^{(2)}(2,7) \arrow[dll] & \mathcal{R}^{(2)}(2,5) \arrow[d] & & \mathcal{R}^{(2)}(2,5) \arrow[dll] \\ \mathcal{R}^{(2)}(0,1)_5 & & \mathcal{R}^{(2)}(0,1)_5 \arrow[ll] \arrow[u] \arrow[ull] & \mathcal{R}^{(2)}(0,1)_7 & & \mathcal{R}^{(2)}(0,1)_7 \arrow[ll] \arrow[u] \arrow[ull] \end{tikzcd}$$

$$\mathcal{R}^{(3)}(0,1,2,5) : \quad \begin{tikzcd} \mathcal{R}^{(2)}(0,2)_7 \arrow[d] & & \mathcal{R}^{(2)}(0,2)_7 \arrow[dll] & \mathcal{R}^{(2)}(0,1)_7 \arrow[d] & & \mathcal{R}^{(2)}(0,1)_7 \arrow[dll] \\ \mathcal{R}^{(2)}(1,5) & & \mathcal{R}^{(2)}(1,5) \arrow[ll] \arrow[u] \arrow[ull] & \mathcal{R}^{(2)}(2,5) & & \mathcal{R}^{(2)}(2,5) \arrow[ll] \arrow[u] \arrow[ull] \end{tikzcd}$$

$$\mathcal{R}^{(3)}(0,0,2,2) : \quad \begin{tikzcd} \mathcal{R}^{(2)}(1,5) \arrow[d] & & \mathcal{R}^{(2)}(1,5) \arrow[dll] & \mathcal{R}^{(2)}(1,7) \arrow[d] & & \mathcal{R}^{(2)}(1,7) \arrow[dll] \\ \mathcal{R}^{(2)}(0,2)_7 & & \mathcal{R}^{(2)}(0,2)_7 \arrow[ll] \arrow[u] \arrow[ull] & \mathcal{R}^{(2)}(0,2)_5 & & \mathcal{R}^{(2)}(0,2)_5 \arrow[ll] \arrow[u] \arrow[ull] \end{tikzcd}$$

$$\mathcal{R}^{(3)}(0,1,2,7) : \quad \begin{tikzcd} \mathcal{R}^{(2)}(0,1)_5 \arrow[d] & & \mathcal{R}^{(2)}(0,1)_5 \arrow[dll] & \mathcal{R}^{(2)}(0,2)_5 \arrow[d] & & \mathcal{R}^{(2)}(0,2)_5 \arrow[dll] \\ \mathcal{R}^{(2)}(2,7) & & \mathcal{R}^{(2)}(2,7) \arrow[ll] \arrow[u] \arrow[ull] & \mathcal{R}^{(2)}(1,7) & & \mathcal{R}^{(2)}(1,7) \arrow[ll] \arrow[u] \arrow[ull] \end{tikzcd}$$

The embedding diagram of the conjugate representation is obtained from these diagrams by reversing all arrows. It is easy to see that at least the embedding diagrams of all rank 2 and rank 3 representations are self-conjugate.

Homomorphisms. Using the relation

$$\mathrm{Hom}(U,V)\cong \mathrm{Hom}(U\otimes V^*,\mathcal{W}^*),\tag{A.5}$$

one can determine the dimension of $\mathrm{Hom}(U,V)$ from $\mathrm{Hom}(-,\mathcal{W}^*)$. One finds that

$$\dim \mathrm{Hom}(U,\mathcal{W}^*)=\begin{cases}1: U\in\{\mathcal{W}(0),\mathcal{W},\mathcal{W}^*,\mathcal{Q},\\ \quad\mathcal{R}^{(2)}(0,2)_5,\mathcal{R}^{(2)}(0,2)_7,\mathcal{R}^{(3)}(0,0,2,2)\}\\ 0: \text{else.}\end{cases}\tag{A.6}$$

For example, $\dim\mathrm{Hom}(U,\mathcal{W}(0))=1$ for $U\in\{\mathcal{W}(0),\mathcal{W},\mathcal{Q}\}$ and $\dim\mathrm{Hom}(U,\mathcal{W}(0))=0$ for all other representations in (1.4) and (1.5).

As an application, let us show that if $\mathcal{W}(0)$ has a projective cover at all, it is not one of the representations listed in (1.4) and (1.5). Recall that a representation P is projective if, given an intertwiner $f:P\to V$, for any surjective $\pi:U\to V$ we can find a (typically non-unique) intertwiner $g:P\to U$ such that $f=\pi\circ g$. We can now check that none of $\mathcal{W}(0),\mathcal{W},\mathcal{Q}$ are projective. To see this, note that

$$\dim\mathrm{Hom}(\mathcal{W}(0),\mathcal{W})=0,\qquad \dim\mathrm{Hom}(\mathcal{Q},\mathcal{W})=0,\qquad \dim\mathrm{Hom}(\mathcal{W},\mathcal{Q})=0.\tag{A.7}$$

If $\mathcal{Q}$ were projective, then for the non-zero morphism $f:\mathcal{Q}\to\mathcal{W}(0)$ and the surjection $\pi:\mathcal{W}\to\mathcal{W}(0)$ we would have to find $g:\mathcal{Q}\to\mathcal{W}$ such that $f=\pi\circ g$. But there is no non-zero intertwiner $g:\mathcal{Q}\to\mathcal{W}$, so this is not possible. Replacing $\mathcal{Q}$ by $\mathcal{W}(0)$ shows that $\mathcal{W}(0)$ is not projective. For $\mathcal{W}$, one can consider $f:\mathcal{W}\to\mathcal{W}(0)$ and $\pi:\mathcal{Q}\to\mathcal{W}(0)$.

A.4. The complete list of fusion rules

The action of $\mathcal{W}(0)$. The fusion product of $\mathcal{W}(0)$ with everything is zero with the exception of

$$\mathcal{W}(0)\otimes\mathcal{W}(0)=\mathcal{W}(0),\qquad \mathcal{W}(0)\otimes\mathcal{W}=\mathcal{W}(0),\qquad \mathcal{W}(0)\otimes\mathcal{Q}=\mathcal{W}(0).\tag{A.8}$$

The action of $\mathcal{W},\mathcal{W}^*,\mathcal{W}(2)$ and $\mathcal{Q},\mathcal{Q}^*,\mathcal{W}(1)$. The representation $\mathcal{W}$ is the vertex operator algebra and acts as the identity in all fusion products. The fusion with the representations $\mathcal{W}^*$ and $\mathcal{W}(2)$ acts as the identity on all representations in (1.4) and (1.5) that are not in grey boxes; on the representations in grey boxes, the fusion is explicitly given as

		Factors	Fusion product			Factors	Fusion product
$\mathcal{W}^*$	$\otimes$	$\mathcal{W}^*$	$\mathcal{W}^*$	$\mathcal{W}(2)$	$\otimes$	$\mathcal{W}^*$	$\mathcal{W}^*$
	$\otimes$	$\mathcal{Q}$	$\mathcal{Q}^*$		$\otimes$	$\mathcal{Q}$	$\mathcal{W}(1)$
	$\otimes$	$\mathcal{Q}^*$	$\mathcal{Q}^*$		$\otimes$	$\mathcal{Q}^*$	$\mathcal{Q}^*$
	$\otimes$	$\mathcal{W}(0)$	0		$\otimes$	$\mathcal{W}(0)$	0
	$\otimes$	$\mathcal{W}(1)$	$\mathcal{Q}^*$		$\otimes$	$\mathcal{W}(1)$	$\mathcal{Q}^*$
	$\otimes$	$\mathcal{W}(2)$	$\mathcal{W}^*$		$\otimes$	$\mathcal{W}(2)$	$\mathcal{W}^*$
	$\otimes$	$\mathcal{W}(5)$	$\mathcal{W}(5)$		$\otimes$	$\mathcal{W}(5)$	$\mathcal{W}(5)$
	$\otimes$	$\mathcal{W}(7)$	$\mathcal{W}(7)$		$\otimes$	$\mathcal{W}(7)$	$\mathcal{W}(7)$

Similarly $\mathcal{Q}$, $\mathcal{Q}^*$ and $\mathcal{W}(1)$ have the same fusion rules with all representations that are not in grey boxes, and the fusion rules of $\mathcal{W}(1)$ are explicitly given below; on the representations in grey boxes, the fusion rules of $\mathcal{Q}$ and $\mathcal{Q}^*$ are

		Factors	Fusion product			Factors	Fusion product
$\mathcal{Q}$	$\otimes$	$\mathcal{Q}$	$\mathcal{W} \oplus \mathcal{W}\left(\frac{1}{3}\right)$	$\mathcal{Q}^*$	$\otimes$	$\mathcal{Q}$	$\mathcal{W}^* \oplus \mathcal{W}\left(\frac{1}{3}\right)$
	$\otimes$	$\mathcal{Q}^*$	$\mathcal{W}^* \oplus \mathcal{W}\left(\frac{1}{3}\right)$		$\otimes$	$\mathcal{Q}^*$	$\mathcal{W}^* \oplus \mathcal{W}\left(\frac{1}{3}\right)$
	$\otimes$	$\mathcal{W}(0)$	$\mathcal{W}(0)$		$\otimes$	$\mathcal{W}(0)$	0
	$\otimes$	$\mathcal{W}(1)$	$\mathcal{W}(2) \oplus \mathcal{W}\left(\frac{1}{3}\right)$		$\otimes$	$\mathcal{W}(1)$	$\mathcal{W}^* \oplus \mathcal{W}\left(\frac{1}{3}\right)$
	$\otimes$	$\mathcal{W}(5)$	$\mathcal{W}(7) \oplus \mathcal{W}\left(\frac{10}{3}\right)$		$\otimes$	$\mathcal{W}(5)$	$\mathcal{W}(7) \oplus \mathcal{W}\left(\frac{10}{3}\right)$
	$\otimes$	$\mathcal{W}(7)$	$\mathcal{W}(5)$		$\otimes$	$\mathcal{W}(7)$	$\mathcal{W}(5)$

The action of $\mathcal{W}(7)$. The simple current $\mathcal{W}(7)$ squares to $\mathcal{W}^*$, and with the exception of $\mathcal{W}(0), \mathcal{W}(1), \mathcal{W}(2), \mathcal{W}$ and $\mathcal{Q}$, the fusion rules organize themselves into $\mathcal{W}(7)$-pairs. The fusion of $\mathcal{W}(7)$ with these special representations is

$$\begin{array}{ll} \mathcal{W}(7) \otimes \mathcal{W}(0) = 0 & \mathcal{W}(7) \otimes \mathcal{W}(1) = \mathcal{W}(5) \\ \mathcal{W}(7) \otimes \mathcal{W}(2) = \mathcal{W}(7) & \mathcal{W}(7) \otimes \mathcal{W} = \mathcal{W}(7) \\ \mathcal{W}(7) \otimes \mathcal{Q} = \mathcal{W}(5), & \end{array} \tag{A.9}$$

while on the remaining representations we have

$$\begin{array}{rcl||rcl} \mathcal{W}^* & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}(7) & \mathcal{Q}^* & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}(5) \\ \mathcal{W}\left(\frac{1}{3}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}\left(\frac{10}{3}\right) & \mathcal{W}\left(\frac{5}{8}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}\left(\frac{33}{8}\right) \\ \mathcal{W}\left(\frac{1}{8}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}\left(\frac{21}{8}\right) & \mathcal{W}\left(\frac{-1}{24}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{W}\left(\frac{35}{24}\right) \\ \mathcal{R}^{(2)}(0,2)_5 & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}(1,7) & \mathcal{R}^{(2)}(0,1)_7 & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}(2,5) \\ \mathcal{R}^{(2)}(0,2)_7 & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}(2,7) & \mathcal{R}^{(2)}(0,1)_5 & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}(1,5) \\ \mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right) & \mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right) \\ \mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) & \mathcal{R}^{(3)}(0,0,1,1) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(3)}(0,1,2,5) \\ \mathcal{R}^{(3)}(0,0,2,2) & \overset{\mathcal{W}(7)}{\longleftrightarrow} & \mathcal{R}^{(3)}(0,1,2,7) & & & \end{array}$$

We only list the fusion products for the first representative of each $\mathcal{W}(7)$ pair. To obtain the fusion of, for example, $\mathcal{W}\left(\frac{5}{8}\right)$ and $\mathcal{W}\left(\frac{21}{8}\right)$, one computes

$$\mathcal{W}\left(\tfrac{5}{8}\right) \otimes \mathcal{W}\left(\tfrac{21}{8}\right) = \mathcal{W}(7) \otimes \mathcal{W}\left(\tfrac{5}{8}\right) \otimes \mathcal{W}\left(\tfrac{1}{8}\right) = \mathcal{W}(7) \otimes \mathcal{R}^{(2)}(0,1)_5 = \mathcal{R}^{(2)}(1,5). \tag{A.10}$$

The remaining products.

		Factors	Fusion product
$\mathcal{W}(1)$	$\otimes$	$\mathcal{W}(1)$	$\mathcal{W}^* \oplus \mathcal{W}\left(\frac{1}{3}\right)$
	$\otimes$	$\mathcal{W}\left(\frac{1}{3}\right)$	$\mathcal{R}^{(2)}(0,1)_7$
	$\otimes$	$\mathcal{W}\left(\frac{5}{8}\right)$	$\mathcal{W}\left(\frac{1}{8}\right)$
	$\otimes$	$\mathcal{W}\left(\frac{1}{8}\right)$	$\mathcal{W}\left(\frac{5}{8}\right) \oplus \mathcal{W}\left(\frac{-1}{24}\right)$
	$\otimes$	$\mathcal{W}\left(\frac{-1}{24}\right)$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$2\mathcal{W}\left(\frac{1}{3}\right) \oplus \mathcal{R}^{(2)}(0,2)_5$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$2\mathcal{W}\left(\frac{10}{3}\right) \oplus \mathcal{R}^{(2)}(0,1)_7$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(2)}(0,1)_5$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$\mathcal{R}^{(2)}(0,2)_7 \oplus \mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$2\mathcal{W}\left(\frac{35}{24}\right) \oplus \mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$2\mathcal{W}\left(\frac{-1}{24}\right) \oplus \mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus \mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus \mathcal{R}^{(3)}(0,0,1,1)$
$\mathcal{W}\left(\frac{1}{3}\right)$	$\otimes$	$\mathcal{W}\left(\frac{1}{3}\right)$	$\mathcal{W}\left(\frac{1}{3}\right) \oplus \mathcal{R}^{(2)}(0,2)_5$
	$\otimes$	$\mathcal{W}\left(\frac{5}{8}\right)$	$\mathcal{W}\left(\frac{-1}{24}\right)$
	$\otimes$	$\mathcal{W}\left(\frac{1}{8}\right)$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{W}\left(\frac{-1}{24}\right)$	$\mathcal{W}\left(\frac{-1}{24}\right) \oplus \mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$2\mathcal{W}\left(\frac{10}{3}\right) \oplus 2\mathcal{R}^{(2)}(0,1)_7$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$2\mathcal{W}\left(\frac{1}{3}\right) \oplus 2\mathcal{R}^{(2)}(2,5)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$2\mathcal{W}\left(\frac{-1}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$2\mathcal{W}\left(\frac{35}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus \mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,1,2,5)$

$\mathcal{W}\big(\frac{5}{8}\big)$	$\otimes$	$\mathcal{W}\big(\frac{5}{8}\big)$	$\mathcal{R}^{(2)}(0,2)_7$
	$\otimes$	$\mathcal{W}\big(\frac{1}{8}\big)$	$\mathcal{R}^{(2)}(0,1)_5$
	$\otimes$	$\mathcal{W}\big(\frac{-1}{24}\big)$	$\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$2\mathcal{W}\big(\frac{5}{8}\big)\oplus 2\mathcal{W}\big(\frac{33}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{W}\big(\frac{1}{8}\big)\oplus 2\mathcal{W}\big(\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)$	$\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)$	$2\mathcal{W}\big(\frac{-1}{24}\big)\oplus 2\mathcal{W}\big(\frac{35}{24}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{33}{8}\big)$
$\mathcal{W}\big(\frac{1}{8}\big)$	$\otimes$	$\mathcal{W}\big(\frac{1}{8}\big)$	$\mathcal{R}^{(2)}(0,2)_7\oplus\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)$
	$\otimes$	$\mathcal{W}\big(\frac{-1}{24}\big)$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$2\mathcal{W}\big(\frac{-1}{24}\big)\oplus\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$2\mathcal{W}\big(\frac{35}{24}\big)\oplus\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$2\mathcal{W}\big(\frac{1}{8}\big)\oplus 2\mathcal{W}\big(\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{W}\big(\frac{5}{8}\big)\oplus 2\mathcal{W}\big(\frac{33}{8}\big)\oplus 2\mathcal{W}\big(\frac{-1}{24}\big)\oplus 2\mathcal{W}\big(\frac{35}{24}\big)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)$	$2\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{10}{3}\big)\oplus\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$	$2\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)\oplus\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)$	$2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{W}\big(\frac{-1}{24}\big)\oplus 4\mathcal{W}\big(\frac{35}{24}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{33}{8}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{W}\big(\frac{-1}{24}\big)\oplus 4\mathcal{W}\big(\frac{35}{24}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
$\mathcal{W}\big(\frac{-1}{24}\big)$	$\otimes$	$\mathcal{W}\big(\frac{-1}{24}\big)$	$\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)\oplus\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$2\mathcal{W}\big(\frac{35}{24}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$2\mathcal{W}\big(\frac{-1}{24}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$2\mathcal{W}\big(\frac{-1}{24}\big)\oplus 2\mathcal{W}\big(\frac{35}{24}\big)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)$	$2\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)\oplus 2\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)$	$2\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{10}{3}\big)\oplus 2\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}\big(\frac{1}{3},\frac{1}{3}\big)$	$2\mathcal{W}\big(\frac{-1}{24}\big)\oplus 2\mathcal{W}\big(\frac{35}{24}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{5}{8}\big)\oplus 2\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{33}{8}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{W}\big(\frac{-1}{24}\big)\oplus 4\mathcal{W}\big(\frac{35}{24}\big)\oplus 4\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 4\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{W}\big(\frac{-1}{24}\big)\oplus 4\mathcal{W}\big(\frac{35}{24}\big)\oplus 4\mathcal{R}^{(2)}\big(\frac{1}{8},\frac{1}{8}\big)\oplus 4\mathcal{R}^{(2)}\big(\frac{5}{8},\frac{21}{8}\big)$

$\mathcal{R}^{(2)}(0,1)_7$	$\otimes$	$\mathcal{R}^{(2)}(0,1)_7$	$4\mathcal{W}\left(\frac{1}{3}\right) \oplus 2\mathcal{R}^{(2)}(2,5) \oplus 2\mathcal{R}^{(2)}(0,2)_5$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$4\mathcal{W}\left(\frac{10}{3}\right) \oplus 2\mathcal{R}^{(2)}(0,1)_7 \oplus 2\mathcal{R}^{(2)}(1,7)$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus \mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$4\mathcal{W}\left(\frac{35}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,1,2,5) \oplus 2\mathcal{R}^{(2)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\mathcal{R}^{(2)}(0,1,2,7)$
$\mathcal{R}^{(2)}(0,2)_5$	$\otimes$	$\mathcal{R}^{(2)}(0,2)_5$	$4\mathcal{W}\left(\frac{1}{3}\right) \oplus 2\mathcal{R}^{(2)}(2,5) \oplus 2\mathcal{R}^{(2)}(0,2)_5$
	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus \mathcal{R}^{(3)}(0,0,1,1)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$4\mathcal{W}\left(\frac{35}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\mathcal{R}^{(2)}(0,1,2,7)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 2\mathcal{R}^{(3)}(0,1,2,5) \oplus 2\mathcal{R}^{(2)}(0,0,2,2)$
$\mathcal{R}^{(2)}(0,2)_7$	$\otimes$	$\mathcal{R}^{(2)}(0,2)_7$	$2\mathcal{R}^{(2)}(0,2)_7 \oplus 2\mathcal{R}^{(2)}(2,7)$
	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{R}^{(2)}(0,1)_5 \oplus 2\mathcal{R}^{(2)}(1,5)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$2\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$2\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\mathcal{R}^{(3)}(0,1,2,7)$
$\mathcal{R}^{(2)}(0,1)_5$	$\otimes$	$\mathcal{R}^{(2)}(0,1)_5$	$2\mathcal{R}^{(2)}(0,2)_7 \oplus 2\mathcal{R}^{(2)}(2,7) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$
			$\oplus 2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right) \oplus 4\mathcal{W}\left(\frac{35}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right) \oplus 4\mathcal{W}\left(\frac{35}{24}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right) \oplus 2\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$2\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
			$\oplus 2\mathcal{R}^{(3)}(0,0,2,2) \oplus 2\mathcal{R}^{(3)}(0,1,2,7)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right) \oplus 4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
			$\oplus 2\mathcal{R}^{(3)}(0,0,1,1) \oplus 2\mathcal{R}^{(3)}(0,1,2,5)$

$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 2\mathcal{R}^{(3)}(0,1,2,5)\oplus 2\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)\oplus 2\mathcal{R}^{(3)}(0,0,1,1)\oplus 2\mathcal{R}^{(3)}(0,1,2,7)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right)\oplus 4\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$8\mathcal{W}\left(\frac{-1}{24}\right)\oplus 8\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
			$\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$8\mathcal{W}\left(\frac{-1}{24}\right)\oplus 8\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
			$\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 2\mathcal{R}^{(3)}(0,1,2,5)\oplus 2\mathcal{R}^{(3)}(0,0,2,2)$
	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$4\mathcal{W}\left(\frac{-1}{24}\right)\oplus 4\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$8\mathcal{W}\left(\frac{-1}{24}\right)\oplus 8\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
			$\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$8\mathcal{W}\left(\frac{-1}{24}\right)\oplus 8\mathcal{W}\left(\frac{35}{24}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{5}{8}\right)$
			$\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{33}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{8},\frac{1}{8}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{5}{8},\frac{21}{8}\right)$
$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$\otimes$	$\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)$	$2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 2\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
			$\oplus 2\mathcal{R}^{(3)}(0,0,2,2)\oplus 2\mathcal{R}^{(3)}(0,1,2,7)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
			$\oplus 4\mathcal{R}^{(3)}(0,0,1,1)\oplus 4\mathcal{R}^{(3)}(0,1,2,5)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 4\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)$
			$\oplus 4\mathcal{R}^{(3)}(0,0,1,1)\oplus 4\mathcal{R}^{(3)}(0,1,2,5)$
$\mathcal{R}^{(3)}(0,0,1,1)$	$\otimes$	$\mathcal{R}^{(3)}(0,0,1,1)$	$8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)\oplus 4\mathcal{R}^{(3)}(0,0,1,1)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,5)\oplus 4\mathcal{R}^{(3)}(0,0,2,2)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,7)$
	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)\oplus 4\mathcal{R}^{(3)}(0,0,1,1)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,5)\oplus 4\mathcal{R}^{(3)}(0,0,2,2)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,7)$
$\mathcal{R}^{(3)}(0,0,2,2)$	$\otimes$	$\mathcal{R}^{(3)}(0,0,2,2)$	$8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{1}{3}\right)\oplus 8\mathcal{R}^{(2)}\left(\frac{1}{3},\frac{10}{3}\right)\oplus 4\mathcal{R}^{(3)}(0,0,1,1)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,5)\oplus 4\mathcal{R}^{(3)}(0,0,2,2)$
			$\oplus 4\mathcal{R}^{(3)}(0,1,2,7)$

Appendix B. Some technical lemmas and proofs

B.1. Associativity for composition of internal Homs

Lemma B.1. *For $g : B \to [U,V]$, we have $g = \phi_B^{(U,V)}(ev_{U,V}\circ(g\otimes \mathrm{id}_U))$.*

Proof. This is a consequence of applying (3.10) for $X = [U,V]$, $Y = B$ and $t = ev_{U,V}$. $\square$

Lemma B.2. $ev_{V,W}\circ(\mathrm{id}_{[V,W]}\otimes ev_{U,V})\circ\alpha^{-1}_{[V,W],[U,V],U} = ev_{U,W}\circ(m_{W,V,U}\otimes \mathrm{id}_U)$.

Proof. First, apply $\phi^{(U,W)}_{[V,W]\otimes[U,V]}$ to both sides. The next step is to show that the resulting morphisms are equal. By definition in (3.12), the left-hand side is equal to $m_{W,V,U}$. On the right-hand side, one uses lemma B.1 with $g = m_{W,V,U}$. □

Lemma B.3. $ev_{U,U} \circ (\eta_U \otimes \mathrm{id}_U) = \lambda_U$.

Proof. From lemma B.1 with $g = \eta_U$, we get $\eta_U = \phi^{(U,U)}_{\mathbf{1}}(ev_{U,U} \circ (\eta_U \otimes \mathrm{id}_U))$. By definition, $\eta_U = \phi^{(U,U)}_{\mathbf{1}}(\lambda_U)$, and the statement follows. □

Proof of theorem 3.2.

Associativity. Let g be the left-hand side of the associativity equation and g' the right-hand side. By lemma B.1, it is enough to show that $ev_{A,D} \circ (g \otimes \mathrm{id}_A) = ev_{A,D} \circ (g' \otimes \mathrm{id}_A)$. For the two sides, one finds (omitting the indices (exercise: add the indices) and using lemma B.2)

$$\begin{aligned} ev \circ (g \otimes \mathrm{id}) &= ev \circ (m \otimes \mathrm{id}) \circ ((\mathrm{id} \otimes m) \otimes \mathrm{id}) = ev \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} \circ ((\mathrm{id} \otimes m) \otimes \mathrm{id}) \\ &= ev \circ [\mathrm{id} \otimes (ev \circ (m \otimes \mathrm{id}))] \circ \alpha^{-1} = ev \circ [\mathrm{id} \otimes (ev \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1})] \circ \alpha^{-1} \\ &= ev \circ (\mathrm{id} \otimes ev) \circ (\mathrm{id} \otimes (\mathrm{id} \otimes ev)) \circ (\mathrm{id} \otimes \alpha^{-1}) \circ \alpha^{-1} \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} ev \circ (g' \otimes \mathrm{id}) &= ev \circ (m \otimes \mathrm{id}) \circ ((m \otimes \mathrm{id}) \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \\ &= ev \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} \circ ((m \otimes \mathrm{id}) \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \\ &= ev \circ (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} \circ (\alpha \otimes \mathrm{id}) \\ &= ev \circ (\mathrm{id} \otimes ev) \circ (\mathrm{id} \otimes (\mathrm{id} \otimes ev)) \circ \alpha^{-1} \circ \alpha^{-1} \circ (\alpha \otimes \mathrm{id}). \end{aligned} \tag{B.2}$$

The two expressions are equal if (putting the indices back)

$$\begin{aligned} &(\mathrm{id}_{[C,D]} \otimes \alpha^{-1}_{[B,C],[A,B],A}) \circ \alpha^{-1}_{[C,D],[B,C]\otimes[A,B],A} \\ &\quad = \alpha^{-1}_{[C,D],[B,C],[A,B]\otimes A} \circ \alpha^{-1}_{[C,D]\otimes[B,C],[A,B],A} \circ (\alpha_{[C,D],[B,C],[A,B]} \otimes \mathrm{id}_A). \end{aligned} \tag{B.3}$$

This equality holds because any two ways of rebracketing are equal in a tensor category. Concretely, it follows from the pentagon equation satisfied by the associator that

$$\alpha_{T\otimes U,V,W} \circ \alpha_{T,U,V\otimes W} = (\alpha_{T,U,V} \otimes \mathrm{id}_W) \circ \alpha_{T,U\otimes V,W} \circ (\mathrm{id}_T \otimes \alpha_{U,V,W}). \tag{B.4}$$

Unit. For the first of the two unit conditions, set $g = m_{B,B,A} \circ (\eta_B \otimes \mathrm{id}_{[A,B]})$ and $g' = \lambda_{[A,B]}$. By lemma B.1, it is enough to show that $ev \circ (g \otimes \mathrm{id}) = ev \circ (g' \otimes \mathrm{id})$. Using also lemma B.3, we get

$$\begin{aligned} ev \circ (g \otimes \mathrm{id}) &= ev \circ (m \otimes \mathrm{id}) \circ ((\eta \otimes \mathrm{id}) \otimes \mathrm{id}) = ev \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} \circ ((\eta \otimes \mathrm{id}) \otimes \mathrm{id}) \\ &= ev \circ (\eta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} = \lambda \circ (\mathrm{id} \otimes ev) \circ \alpha^{-1} = ev \circ \lambda \circ \alpha^{-1}. \end{aligned} \tag{B.5}$$

This is equal to $ev \circ (g' \otimes \mathrm{id})$ if $\lambda_{[A,B]\otimes A} \circ \alpha^{-1}_{\mathbf{1},[A,B],A} = \lambda_{[A,B]} \otimes \mathrm{id}_A$. The last identity follows from the axioms of a tensor category; see [56, proposition 1.1]. For the second unit condition, set $g = m_{B,B,A} \circ (\mathrm{id}_{[A,B]} \otimes \eta_A)$ and $g' = \rho_{[A,B]}$. We get

$$ev \circ (g \otimes \mathrm{id}) = ev \circ (m \otimes \mathrm{id}) \circ ((\mathrm{id} \otimes \eta) \otimes \mathrm{id}) = ev \circ (\mathrm{id} \otimes \lambda) \circ \alpha^{-1}. \tag{B.6}$$

For this to be equal to $ev \circ (g' \otimes \mathrm{id})$, we need $(\mathrm{id}_{[A,B]} \otimes \lambda_A) \circ \alpha^{-1}_{[A,B],\mathbf{1},A} = \rho_{[A,B]} \otimes \mathrm{id}_A$, which is an instance of the triangle condition

$$\mathrm{id}_U \otimes \lambda_V = (\rho_U \otimes \mathrm{id}_V) \circ \alpha_{U,\mathbf{1},V}. \tag{B.7}$$

B.2. Proof of theorems 3.9 and 3.10

Lemma B.4. *Let $\mathcal{C}$ be a tensor category satisfying condition* C. *If $U \in \mathcal{C}^b$, then so is U^*.*

Proof. The duality morphisms for U^* are constructed from δ_U and the duality morphisms of U as

$$\begin{aligned} b_{U^*} &= \mathbf{1} \xrightarrow{\tilde{b}_U} U^* \otimes U \xrightarrow{\mathrm{id}_{U^*}\otimes\delta_U} U^* \otimes U^{**}, \\ d_{U^*} &= U^{**} \otimes U^* \xrightarrow{\delta_U^{-1}\otimes\mathrm{id}_{U^*}} U \otimes U^* \xrightarrow{\tilde{d}_U} \mathbf{1}, \\ \tilde{b}_{U^*} &= \mathbf{1} \xrightarrow{b_U} U \otimes U^* \xrightarrow{\delta_U\otimes\mathrm{id}_{U^*}} U^{**} \otimes U^*, \\ \tilde{d}_{U^*} &= U^* \otimes U^{**} \xrightarrow{\mathrm{id}_{U^*}\otimes\delta_U^{-1}} U^* \otimes U \xrightarrow{d_U} \mathbf{1}. \end{aligned} \tag{B.8}$$

The check that these satisfy the duality properties is a straightforward calculation using the duality properties of $b_U, d_U, \tilde{b}_U, \tilde{d}_U$. It is also clear that $b_{U^*}, \tilde{b}_{U^*}$ are injective because they are the composition of an injective map and a bijection. □

Lemma B.5. *Let $\mathcal{C}$ be a tensor category.*

(i) Let $U \in \mathcal{C}$ have a right dual. Then $b_U : \mathbf{1} \to U \otimes U^\vee$ is injective if and only if the map $f \mapsto f \otimes \mathrm{id}_U : \mathrm{Hom}(X, \mathbf{1}) \to (X \otimes U, \mathbf{1} \otimes U)$ is injective for all $X \in \mathcal{C}$.
(ii) Let $U \in \mathcal{C}$ have a left dual. Then $\tilde{b}_U : \mathbf{1} \to {}^\vee U \otimes U$ is injective if and only if the map $f \mapsto \mathrm{id}_U \otimes f : \mathrm{Hom}(X, \mathbf{1}) \to (U \otimes X, U \otimes \mathbf{1})$ is injective for all $X \in \mathcal{C}$.

Proof. The proof is straightforward. For example, for (i) one shows with the help of the duality morphisms that $b_U \circ f = b_U \circ g$ is equivalent to $f \otimes \mathrm{id}_U = g \otimes \mathrm{id}_U$. □

Lemma B.6. *Let $\mathcal{C}$ be a tensor category satisfying condition* C. *If $U, V \in \mathcal{C}^b$, then so is $U \otimes V$.*

Proof. The duality morphisms for $U \otimes V$ are constructed as in the proof of lemma 3.7, together with the observation (3.19) that for $U, V \in \mathcal{C}^b$ we have $(U \otimes V)^* \cong V^* \otimes U^*$. It remains to check that $b_{U\otimes V}$ and $\tilde{b}_{U\otimes V}$ are injective. This follows from lemma B.5; let us go through the argument for $b_{U\otimes V} : \mathbf{1} \to (U \otimes V) \otimes (U \otimes V)^*$. This morphism is injective if and only if the map $f \mapsto f \otimes \mathrm{id}_{U\otimes V}$ is injective. But by assumption b_U and b_V are injective so that, again by lemma B.5,

$$f \otimes \mathrm{id}_{U\otimes V} = 0 \quad \Rightarrow \quad (f \otimes \mathrm{id}_U) \otimes \mathrm{id}_V = 0 \quad \Rightarrow \quad f \otimes \mathrm{id}_U = 0 \quad \Rightarrow \quad f = 0. \tag{B.9}$$

Thus, $b_{U\otimes V}$ is injective. □

Proof of theorem 3.9. Part (i) amounts to lemma B.4 and part (ii) to lemma B.6. □

As in (3.9), we call a morphism $p : U \otimes V \to \mathbf{1}^*$ *non-degenerate* if $\pi_{U,V}^{-1}(p) : U \to V^*$ is an isomorphism.

Lemma B.7. *Let $\mathcal{C}$ be an Abelian tensor category satisfying condition* C. *For a morphism $p : U \otimes V \to \mathbf{1}^*$, the following are equivalent.*

(i) p is non-degenerate.
(ii) For all $X, Y \in \mathcal{C}$ and all $f : X \to U$, $g : Y \to V$ we have that $p \circ (f \otimes \mathrm{id}_V) = 0$ implies $f = 0$ and $p \circ (\mathrm{id}_U \otimes g) = 0$ implies $g = 0$.

Proof. Let f and g be as in part (ii). Since $\pi_{U,V}$ is natural in U and V, so is $\pi_{U,V}^{-1}$. This in turn means that $\pi_{X,V}^{-1}(p \circ (f \otimes \mathrm{id}_V)) = \pi_{U,V}^{-1}(p) \circ f$ and $\pi_{U,Y}^{-1}(p \circ (\mathrm{id}_U \otimes g)) = g^* \circ \pi_{U,V}^{-1}(p)$.

(i) (i)⇒(ii): suppose $p \circ (f \otimes \mathrm{id}_V) = 0$. Then $0 = \pi_{X,V}^{-1}(p \circ (f \otimes \mathrm{id}_V)) = \pi_{U,V}^{-1}(p) \circ f$. Since $\pi_{U,V}^{-1}(p)$ is an isomorphism, this implies $f = 0$. That $p \circ (\mathrm{id}_U \otimes g) = 0$ implies $g = 0$ follows in the same way.

(ii) (ii)⇒(i): suppose $\pi_{U,V}^{-1}(p) \circ f = 0$. Then $\pi_{X,V}^{-1}(p \circ (f \otimes \mathrm{id}_V)) = 0$ and consequently $p \circ (f \otimes \mathrm{id}_V) = 0$. By assumption, this implies $f = 0$. Thus, $\pi_{U,V}^{-1}(p)$ is injective. Similarly, $g^* \circ \pi_{U,V}^{-1}(p) = 0$ implies $g = 0$ (and so $g^* = 0$). Thus, $\pi_{U,V}^{-1}(p)$ is surjective. Since $\mathcal{C}$ is Abelian, this implies that $\pi_{U,V}^{-1}(p)$ is an isomorphism. □

Proof of theorem 3.10. Part (i) follows from lemmas B.4 and B.6 because they show that if $A, B \in \mathcal{C}^b$, so is $B \otimes A^*$, and objects in $\mathcal{C}^b$ are necessarily non-zero (otherwise the duality morphism $b_{B\otimes A^*}$ cannot be injective). Part (ii) holds by definition of $\mathcal{C}^b$ because by (3.18), the unit morphism is just $\eta_A = b_A$. Part (iii) can be proved as follows. By (3.18) and (3.20), we can write $\varepsilon_A \circ m_{A,B,A} = \pi_{A,A^*}(\delta_A) \circ (\mathrm{id}_A \otimes d_B \otimes \mathrm{id}_{A^*}) =: p$. We will show that p satisfies condition (ii) of lemma B.7. Let $f : X \to A \otimes B^*$ and suppose that $p \circ (f \otimes \mathrm{id}_{B\otimes A^*}) = 0$. We can write

$$0 = p \circ (f \otimes \mathrm{id}_{B\otimes A^*}) = \varepsilon_A \circ (\tilde{f} \otimes \mathrm{id}_{A^*}) \qquad \text{with} \quad \tilde{f} = (\mathrm{id}_A \otimes d_B) \circ (f \otimes \mathrm{id}_B). \tag{B.10}$$

By definition $\pi_{A,A^*}^{-1}(\varepsilon_A) = \delta_A$, so that ε_A is non-degenerate. By lemma B.7, the above equation implies $\tilde{f} = 0$. Applying the duality morphism b_B to remove d_B shows that then also $f = 0$. The argument that $p \circ (\mathrm{id}_{A\otimes B^*} \otimes g) = 0$ implies $g = 0$ is similar. Thus, p is non-degenerate. □

B.3. The kernel of $\mathbf{b}_U$ *and* $\tilde{\mathbf{b}}_U$

The following lemma provides a method to deduce the kernel of b_U and $\tilde{b}_U$ from the action of the tensor product on objects.

Lemma B.8. *Let $\mathcal{C}$ be an Abelian tensor category and suppose that U has a right and a left dual. Let K be the kernel of b_U and $\tilde{K}$ the kernel of $\tilde{b}_U$.*

(i) $K \otimes U = 0$ and $U^\vee \otimes K = 0$.
(ii) If S is a subobject of $\mathbf{1}$ such that $S \otimes U = 0$ or $U^\vee \otimes S = 0$, then S is a subobject of K.
(iii) $\tilde{K} \otimes {}^\vee U = 0$ and $U \otimes \tilde{K} = 0$.
(iv) If $\tilde{S}$ is a subobject of $\mathbf{1}$ such that $\tilde{S} \otimes {}^\vee U = 0$ or $U \otimes \tilde{S} = 0$, then $\tilde{S}$ is a subobject of $\tilde{K}$.

Proof. Let us prove (i) and (ii) in detail; parts (iii) and (iv) work similarly. We will not write out unit isomorphisms and associators.

(i) Let $k : K \to \mathbf{1}$ be the embedding of the kernel. As in the proof of lemma B.5, applying the duality morphisms to $b_U \circ k = 0$ gives $k \otimes \mathrm{id}_U = 0$. From this, we conclude that

$$\begin{aligned} 0 &= \big(K \otimes U \otimes {}^\vee U \xrightarrow{k \otimes \mathrm{id}_U \otimes \mathrm{id}_{^\vee U}} U \otimes {}^\vee U \xrightarrow{\tilde{d}_U} \mathbf{1}\big) \\ &= \big(K \otimes U \otimes {}^\vee U \xrightarrow{\mathrm{id}_K \otimes \tilde{d}_U} K \xrightarrow{k} \mathbf{1}\big). \end{aligned} \tag{B.11}$$

Since k is injective, it follows that $\mathrm{id}_K \otimes \tilde{d}_U = 0$. Using the left duality morphisms, this in turn implies $\mathrm{id}_{K\otimes U} = 0$, i.e. $K \otimes U = 0$. That $U^\vee \otimes K = 0$ can be seen similarly.

(ii) Let $s : S \to \mathbf{1}$ be the subobject embedding. If $S \otimes U = 0$, then also $s \otimes \mathrm{id}_U = 0$. Again as in the proof of lemma B.5, this implies $b_U \circ s = 0$. Thus, $s : S \to \mathbf{1}$ will factor through K via an injective morphism. The argument starting from $U^\vee \otimes S = 0$ is similar. □

Note that the statement of the lemma cannot be split into two independent statements about right and left duals, because the proof of (i), which is a statement about the right dual, did require the left dual and vice versa for part (iii). The lemma tells us that if U has a right and a left dual, then the kernel of b_U is the maximal subobject S of $\mathbf{1}$ for which $S \otimes U = 0$ and the kernel of $\tilde{b}_U$ is the maximal subobject $\tilde{S}$ of $\mathbf{1}$ for which $U \otimes \tilde{S} = 0$.

Let us now turn again to the $\mathcal{W}_{2,3}$ model. As already mentioned a number of times, we believe that the representations listed in (1.4) and (1.5) which are not in grey boxes have the property that U^* is a right and left dual of U. To check which of these are in $\mathrm{Rep}(\mathcal{W}_{2,3})^b$, it remains to select those U for which b_U and $\tilde{b}_U$ are injective. We will do that with the help of lemma B.8. The only non-trivial subobject of $\mathcal{W}$ is $\mathcal{W}(2)$. From the fusion rules in appendix A.4, we see that $\mathcal{W}(2) \otimes \mathcal{R} \cong \mathcal{R} \neq 0$ for all representations in (1.4) and (1.5) not in grey boxes. Therefore, $\mathcal{W}(2)$ cannot be in the kernel of $b_\mathcal{R}$ or $\tilde{b}_\mathcal{R}$ for any of these $\mathcal{R}$, and so the kernels of $b_\mathcal{R}$ and $\tilde{b}_\mathcal{R}$ are trivial.

References

[1] Jeng M, Piroux G and Ruelle P 2006 Height variables in the Abelian sandpile model: scaling fields and correlations *J. Stat. Mech.* P10015 (arXiv:cond-mat/0609284)

[2] Pearce P A and Rasmussen J 2007 Solvable critical dense polymers *J. Stat. Mech.* P02015 (arXiv:hep-th/0610273)

[3] Read N and Saleur H 2007 Associative-algebraic approach to logarithmic conformal field theories *Nucl. Phys.* B **777** 316 (arXiv:hep-th/0701117)

[4] Ruelle P 2007 Wind on the boundary for the Abelian sandpile model *J. Stat. Mech.* P09013 (arXiv:0707.3766 [cond-mat.stat-mech])

[5] Mathieu P and Ridout D 2007 From percolation to logarithmic conformal field theory *Phys. Lett.* B **657** 120 (arXiv:0708.0802 [hep-th])

[6] Rasmussen J and Pearce P A 2008 W-extended fusion algebra of critical percolation *J. Phys. A: Math. Theor.* **41** 295208 (arXiv:0804.4335 [hep-th])

[7] Ridout D 2009 On the percolation BCFT and the crossing probability of Watts *Nucl. Phys.* B **810** 503 (arXiv:0808.3530 [hep-th])

[8] Saint-Aubin Y, Pearce P A and Rasmussen J 2009 Geometric exponents, SLE and logarithmic minimal models *J. Stat. Mech.* P02028 (arXiv:0809.4806 [cond-mat.stat-mech])

[9] Nigro A 2009 Integrals of motion for critical dense polymers and symplectic fermions arXiv:0903.5051 [hep-th]

[10] Grumiller D and Johansson N 2009 Consistent boundary conditions for cosmological topologically massive gravity at the chiral point *Int. J. Mod. Phys.* D **17** 2367 (arXiv:0808.2575 [hep-th])

[11] Maloney A, Song W and Strominger A 2009 Chiral gravity, log gravity and extremal CFT arXiv:0903.4573 [hep-th]

[12] Gurarie V and Ludwig A W W 2002 Conformal algebras of 2D disordered systems *J. Phys. A: Math. Gen.* **35** L377 (arXiv:cond-mat/9911392)

[13] Gaberdiel M R and Kausch H G 1996 A rational logarithmic conformal field theory *Phys. Lett.* B **386** 131 (arXiv:hep-th/9606050)

[14] Fuchs J, Hwang S, Semikhatov A M and Tipunin I Y 2004 Nonsemisimple fusion algebras and the Verlinde formula *Commun. Math. Phys.* **247** 713 (arXiv:hep-th/0306274)

[15] Carqueville N and Flohr M 2006 Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of W-algebras *J. Phys. A: Math. Gen.* **39** 951 (arXiv:math-ph/0508015)

[16] Gaberdiel M R and Runkel I 2008 From boundary to bulk in logarithmic CFT *J. Phys. A: Math. Theor.* **41** 075402 (arXiv:0707.0388 [hep-th])

[17] Adamovic D and Milas A 2008 On the triplet vertex algebra W(p) *Adv. Math.* **217** 2664 (arXiv:0707.1857 [math.QA])

[18] Feigin B L, Gainutdinov A M, Semikhatov A M and Tipunin I Y 2006 Logarithmic extensions of minimal models: characters and modular transformations *Nucl. Phys.* B **757** 303 (arXiv:hep-th/0606196)

[19] Rasmussen J 2009 W-extended logarithmic minimal models *Nucl. Phys.* B **807** 495 (arXiv:0805.2991 [hep-th])
[20] Eberle H and Flohr M 2006 Virasoro representations and fusion for general augmented minimal models *J. Phys. A: Math. Gen.* **39** 15245 (arXiv:hep-th/0604097)
[21] Rasmussen J 2008 Polynomial fusion rings of W-extended logarithmic minimal models arXiv:0812.1070 [hep-th]
[22] Ruelle P 2002 A $c = -2$ boundary changing operator for the Abelian sandpile model *Phys. Lett.* B **539** 172 (arXiv:hep-th/0203105)
[23] Piroux G and Ruelle P 2004 Pre-logarithmic and logarithmic fields in sandpile model *J. Stat. Mech.* P10005 (arXiv:hep-th/0407143)
[24] Piroux G and Ruelle P 2005 Logarithmic scaling for height variables in the Abelian sandpile model *Phys. Lett.* B **607** 188 (arXiv:cond-mat/0410253)
[25] Izmailian N S, Priezzhev V B, Ruelle P and Hu C K 2005 Logarithmic conformal field theory and boundary effects in the dimer model *Phys. Rev. Lett.* **95** 260602 (arXiv:cond-mat/0512703)
[26] Pearce P A, Rasmussen J and Zuber J-B 2006 Logarithmic minimal models *J. Stat. Mech.* P11017 (arXiv:hep-th/0607232)
[27] Pearce P A, Rasmussen J and Ruelle P 2008 Integrable boundary conditions and W-extended fusion in the logarithmic minimal models LM(1,p) *J. Phys. A: Math. Gen.* **41** 295201 (arXiv:0803.0785 [hep-th])
[28] Creutzig T, Quella T and Schomerus V 2008 Branes in the $GL(1|1)$ WZNW-model *Nucl. Phys.* B **792** 257 (arXiv:0708.0583 [hep-th])
[29] Quella T, Schomerus V and Creutzig T 2008 Boundary spectra in superspace sigma-models *J. High Energy Phys.* JHEP10(2008)024 (arXiv:0712.3549 [hep-th])
[30] Creutzig T and Schomerus V 2009 Boundary correlators in supergroup WZNW models *Nucl. Phys.* B **807** 471 (arXiv:0804.3469 [hep-th])
[31] Creutzig T 2009 Geometry of branes on supergroups *Nucl. Phys.* B **812** 301 (arXiv:0809.0468 [hep-th])
[32] Mitev V, Quella T and Schomerus V 2008 Principal chiral model on superspheres *J. High Energy Phys.* JHEP11(2008)086 (arXiv:0809.1046 [hep-th])
[33] Creutzig T and Ronne P B 2008 The $GL(1|1)$-symplectic fermion correspondence arXiv:0812.2835 [hep-th]
[34] Kogan I I and Wheater J F 2000 Boundary logarithmic conformal field theory *Phys. Lett.* B **486** 353 (arXiv:hep-th/0003184)
[35] Ishimoto Y 2001 Boundary states in boundary logarithmic CFT *Nucl. Phys.* B **619** 415 (arXiv:hep-th/0103064)
[36] Kawai S and Wheater J F 2001 Modular transformation and boundary states in logarithmic conformal field theory *Phys. Lett.* B **508** 203 (arXiv:hep-th/0103197)
[37] Bredthauer A and Flohr M 2002 Boundary states in $c = -2$ logarithmic conformal field theory *Nucl. Phys.* B **639** 450 (arXiv:hep-th/0204154)
[38] Bredthauer A 2003 Boundary states and symplectic fermions *Phys. Lett.* B **551** 378 (arXiv:hep-th/0207181)
[39] Ishimoto Y 2004 Two-point functions and logarithmic boundary operators in boundary logarithmic conformal field theories *J. High Energy Phys.* JHEP08(2004)039 (arXiv:hep-th/0405237)
[40] Gaberdiel M R and Runkel I 2006 The logarithmic triplet theory with boundary *J. Phys. A: Math. Gen.* **39** 14745 (arXiv:hep-th/0608184)
[41] Creutzig T, Quella T and Schomerus V 2008 New boundary conditions for the $c = -2$ ghost system *Phys. Rev.* D **77** 026003 (arXiv:hep-th/0612040)
[42] Gainutdinov A M and Tipunin I Y 2007 Radford, Drinfeld, and Cardy boundary states in (1,p) logarithmic conformal field models arXiv:0711.3430 [hep-th]
[43] Runkel I 1999 Boundary structure constants for the A-series Virasoro minimal models *Nucl. Phys.* B **549** 563 (arXiv:hep-th/9811178)
[44] Felder G, Fröhlich J, Fuchs J and Schweigert C 2002 Correlation functions and boundary conditions in RCFT and three-dimensional topology *Compos. Math.* **131** 189 (arXiv:hep-th/9912239)
[45] Fuchs J, Runkel I and Schweigert C 2002 TFT construction of RCFT correlators: I. Partition functions *Nucl. Phys.* B **646** 353 (arXiv:hep-th/0204148)
[46] Fjelstad J, Fuchs J, Runkel I and Schweigert C 2008 Uniqueness of open/closed rational CFT with given algebra of open states *Adv. Theor. Math. Phys.* **12** 1283 (arXiv:hep-th/0612306)
[47] Kong L and Runkel I 2007 Cardy algebras and sewing constraints: I arXiv:0708.1897 [math.QA]
[48] Kong L and Runkel I 2008 Morita classes of algebras in modular tensor categories *Adv. Math.* **219** 1548 (arXiv:0708.1897 [math.CT])
[49] Cardy J L 1989 Boundary conditions, fusion rules and the Verlinde formula *Nucl. Phys.* B **324** 581
[50] Gaberdiel M R and Kausch H G 1999 A local logarithmic conformal field theory *Nucl. Phys.* B **538** 631 (arXiv:hep-th/9807091)

[51] Adamovic D and Milas A 2009 Lattice construction of logarithmic modules for certain vertex algebras arXiv:0902.3417 [math.QA]
[52] Nahm W 1994 Quasirational fusion products *Int. J. Mod. Phys.* B **8** 3693 (arXiv:hep-th/9402039)
[53] Gaberdiel M R and Kausch H G 1996 Indecomposable fusion products *Nucl. Phys.* B **477** 293 (arXiv:hep-th/9604026)
[54] Fuchs J 2007 On non-semisimple fusion rules and tensor categories *Contemp. Math.* **442** 315 (arXiv:hep-th/0602051)
[55] Wang W 1993 Rationality of Virasoro vertex operator algebras *Int. Math. Res. Not.* **7** 197
[56] Joyal A and Street R 1993 Braided tensor categories *Adv. Math.* **102** 20
[57] Bakalov B and Kirillov A A 2001 *Lectures on Tensor Categories and Modular Functors* (Providence, RI: American Mathematical Society)
[58] Huang Y-Z 1996 Virasoro vertex operator algebras, the (nonmeromorphic) operator product expansion and the tensor product theory *J. Alg.* **182** 201 (arXiv:q-alg/9505020)
[59] Huang Y-Z 2008 Rigidity and modularity of vertex tensor categories *Commun. Contemp. Math.* **10** 871 (arXiv:math/0502533)
[60] Kausch H G 1991 Extended conformal algebras generated by a multiplet of primary fields *Phys. Lett.* B **259** 448
[61] Huang Y Z, Lepowsky J and Zhang L 2007 Logarithmic tensor product theory for generalized modules for a conformal vertex algebra arXiv:0710.2687 [math.QA]
[62] Huang Y-Z 2007 Cofiniteness conditions, projective covers and the logarithmic tensor product theory arXiv:0712.4109 [math.QA]
[63] Huang Y-Z 2009 Representations of vertex operator algebras and braided finite tensor categories arXiv:0903.4233 [math.QA]
[64] Cardy J L 1984 Conformal invariance and surface critical behavior *Nucl. Phys.* B **240** 514
[65] Cardy J L and Lewellen D C 1991 Bulk and boundary operators in conformal field theory *Phys. Lett.* B **259** 274
[66] Lewellen D C 1992 Sewing constraints for conformal field theories on surfaces with boundaries *Nucl. Phys.* B **372** 654
[67] Behrend R E, Pearce P A, Petkova V B and Zuber J-B 2000 Boundary conditions in rational conformal field theories *Nucl. Phys.* B **570** 525
Behrend R E, Pearce P A, Petkova V B and Zuber J-B 2000 Boundary conditions in rational conformal field theories *Nucl. Phys.* B **579** 707 (arXiv:hep-th/9908036)
[68] Semikhatov A M 2007 A note on the logarithmic (p, p') fusion arXiv:0710.5157 [hep-th]
[69] Gaberdiel M R 1994 Fusion in conformal field theory as the tensor product of the symmetry algebra *Int. J. Mod. Phys.* A **9** 4619 (arXiv:hep-th/9307183)
[70] Rohsiepe F 1996 On reducible but indecomposable representations of the Virasoro algebra arXiv:hep-th/9611160
[71] Kytölä K and Ridout D 2009 On staggered indecomposable Virasoro modules arXiv:0905.0108 [math-ph]
[72] Lane S Mac 1998 *Categories for the Working Mathematician* 2nd edn (Berlin: Springer)
[73] Majid S 2000 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[74] Ostrik V 2003 Module categories, weak Hopf algebras and modular invariants *Transform. Groups* **8** 177 (arXiv:math/0111139)
[75] Quella T and Schomerus V 2007 Free fermion resolution of supergroup WZNW models *J. High Energy Phys.* JHEP09(2007)085 (arXiv:0706.0744 [hep-th])